Solving Time-Fractional Korteureg-de-varies Equations by Fractional Reduced Differential Transform Method

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Abstract

In this paper, an analytic solution which has to do with the series expansion approach is proposed to determine the solution of time K-de V equation, specifically by FRDTM. The fractional derivatives are demonstrated in the Caputo sense. We compare the obtained results with R-K fourth order Method. It is possible to obtain solution closed to exact solution of a partial differential equation. To sum it up all, the accuracy, robustness, efficiency and convergence of this techniques are then illustrated through the numerical examples presented in this paper.

Keywords: Korteureg-de-varies Burger; Reduced differential transform method; approximate solution; Runge-Kutta equation.

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1.   Introduction

In recent years fractional differential operators have become the vehicle through which approximate solutions of most linear and nonlinear PDEs is determined. The fast changing world has made fractional partial differential equation to receive the desired attention in the application in diverse fields such as , mathematics, biology, physics, electrical circuit, fluid mechanics, medicine, and many more [5, 9, 14, 15, 16]. It has been established by many researchers that conducting derivatives in non-integer are very robust way of giving a vivid descriptions of many physical phenomena including diffusion process, heat conducting, damping law, rheology. It is important to note that there is no specific method in general that provides an exact solution in terms of fractional equations and hence obtaining approximate solutions is indispensable in science.

Several methods have been explored and applied to obtain solutions to both linear and non-linear fractional equations and among them are: the Adomian Decomposition Method (ADM) [3, 6, 22, 23], the Variational Iteration Method (VIM) [3, 21], the Differential Transform Method (DTM) [4, 12, 13] and the Homotopy Perturbation Method (HPM) [17, 24]. Most of these methods sometimes require a very huge computations in order to obtain approximate solutions. The fractional reduced differential transform method proposed by Keskin and Oturanc [7] is therefore to overcome some of these problems already mentioned in terms of highly complicated computations. For this method, it is able to lead to some cases both exact and approximate solution in a quickly convergent power series. More often, a few numbers of iterations is therefore, required of the series solution for numerical purposes and having high accuracy at the same time [18, 19]. This method is very efficient, can be relied upon and robust analytically [18, 20].

In this paper, we propose an approximate analytical solution of the time fractional partial differential equation of the order α (0<α ≤ 1) in a series form which rapidly converges to exact solution using FRDTM. In Section 2, the basic preliminary on FRDTM and notations on fractional calculus theory have been presented. In Section 3, the preliminary on FRDTM has been provided in detail. In Section 4, the fractional reduced differential transform method has been applied to determine the approximate solutions (FRDTM) of our partial differential equation. Finally, in Section 5 discussion and conclusion of this paper are also presented.

2.   Basic definitions and notations on fractional calculus theory

This section provides some useful notations and definitions that will be utilized in the subsequent sections. The theory of fractional calculus has been around almost the two decades in the literature. Numerous definitions of fractional integrals and derivatives have come up but the first proper definition is attributable to Liouville as follows.

2.1 Definition

A real function $h(x),\ x > 0$ is said to be in the space $C_\mu, \mu \in R$ if there exists a real number $q > \mu$ such that $h(x) = x^q f(x)$ where $h(x) \in C[0, \infty)$, and it is said to be in the space C_μ^m If $h^{(m)} \in C_\mu, m \in N$.

2.2 Definition

For a function $h(x)$, the Riemann - Liouville fractional integral operator [11] of order $\alpha \geq 0$, is expressed as

$$\begin{cases} J^{\alpha}h(x) = \dfrac{1}{\Gamma(\alpha)} \int\limits_0^x (x-t)^{\alpha-1} h(t)dt, \alpha > 0, x > 0 \\ J^0 h(x) = h(x) \end{cases} \tag{2.1}$$

The Riemanne - Liouville derivative possesses particular shortcomings when applying in real life situations in fractional differential sense which calls for the definition of fractional order initial condition, which have no physical meaningful explanation yet. To deal with this obstacle, a modified version of Riemanne - Liouville fractional derivative with operator D^{α} is proposed by Caputo and Mainardi [1]. The Caputo fractional derivative provides room for making use of initial and boundary conditions concerning integer order derivatives, lead to vivid physical meanings.

2.3 Definition

The fractional derivative of h in the Caputo sense [16] can be stated as

$$D^{\alpha}h(x) = J^{m-\alpha}D^m h(x) = \dfrac{1}{\Gamma(m-\alpha)} \int\limits_0^x (x-t)^{m-\alpha-1} h^{(m)}(t)dt, \tag{2.2}$$

for $-1 < \alpha \leq m, m \in N, x > 0, h \in C_{-1}^m$

The major basic characteristics of the Caputo fractional derivative are stated as.

2.4 Lemma

If $m-1 < \alpha \leq m, m \in N$ *and* $h \in C_{\mu}^m, \mu \geq -1$ then

$$\begin{cases} D^{\alpha}J^{\alpha}h(x) = h(x), x > 0, \\ D^{\alpha}J^{\alpha}h(x) = h(x) - \sum\limits_{k=0}^m h^{(k)}(0^+)\dfrac{x^k}{k!}, x > 0, \end{cases} \tag{2.3}$$

In this work, the Caputo fractional derivative is used because it provides avenue to utilize the concept of initial and boundary conditions to be part in the derivative of the problem. For more details about fractional derivatives see [2, 16].

3.   Fractional reduced differential transform method

In this section, we make use of reduced differential transform method for two variable function $u(x,t)$ which has been proposed in [7, 8]. Suppose a function of two variables $u(x,t)$ which is analytic and differentiated continuously in the domain of our interest being studied, and assume that it can be expressed in the form $u(x,t) = h(x)f(t)$.

3.1 Definition

If function $u(x;t)$ is analytic and differentiated continuously with respect to x and t in the domain of interest, then let

$$U_k(x) = \frac{1}{k!}\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}u(x,t)\right]_{t=0}, \tag{3.1}$$

where the t -dimensional spectrum function $U_k(x)$ is referred to as the transformed function which is called Γ function.

The differential inverse transform of $U_k(x)$ is expressed as

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)t^{k\alpha}, \tag{3.2}$$

putting together equations. (3.1) and (3.2) leads to

$$U_k(x) = \sum_{k=0}^{\infty}\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}u(x,t)\right]_{t=0}t^{k\alpha}. \tag{3.3}$$

In real implementation of RDTM and considering $U_O(x) = h(x)$ as transformation of initial condition leads to

$$u(x,0) = h(x), \tag{3.4}$$

By considering equation (3.2), the function $u(x,t)$ can therefore, be approximated by a finite series as

$$\tilde{u}_n(x,t) = \sum_{k=0}^{n} U_k(x)t^{k\alpha} \tag{3.5}$$

A direct iterative computations, leads to the $U_k(x)$ values for $k = 1, 2, ..., n$ .The inverse transformation of the

function $\{U_k(x)\}_{k=0}^{n}$ is then determined which provides the approximation solution as $\tilde{u}_n(x,t)$, where n represents order of approximate solution obtained. Finally, the exact solution is determined by taking limit of the function $u(x,t) = \lim_{n \to \infty} \tilde{u}_n(x,t)$.

We state some of the basic properties of the reduced differential transformation obtained from equations (3.1) and (3.2) which are presented in Table 1.

Table 1: Reduced differential transformations

Functional Form	Transformed Form
$u(x,t)$	$U_k = \dfrac{1}{\Gamma(k\alpha+1)}\left[\dfrac{\partial^{k\alpha}}{\partial^{k\alpha}}u(x,t)\right]_{t=0}$
$w(x,t) = u(x,t) \mp v(x,t)$	$W_k(x) = U_k(x) \mp V_k(x)$
$w(x,t) = cu(x,t)$	$W_k(x) = cU_k(x)$
$w(x,t) = \dfrac{\partial^{N_\alpha}}{\partial t^{N_\alpha}}u(x,t)$	$W_k(x) = \dfrac{\Gamma(k\alpha + N_\alpha +1)}{\Gamma(k\alpha +1))}U_{k+N}(x)$
$w(x,t) = x^m t^n$	$W_k(x) = x^m \sigma(k\alpha - n)$
$w(x,t) = x^m t^n u(x,t)$	$W_k(x) = x^m \sigma U(k\alpha - n)$
$w(x,t) = u(x,t)u(x,t)$	$W_k(x) = \displaystyle\sum_{r=0}^{k} V_r(x)U_{k-r}(x) = \sum_{r=0}^{k} U_r(x)V_{k-r}(x)$
$w(x,t) = \dfrac{\partial^{N\alpha}}{\partial t^{N\alpha}}u(x,t)$	$W_k(x) = \dfrac{\Gamma(k\alpha + N\alpha +1)}{\Gamma(k\alpha +1))}U_{k+N}(x)$
$w(x,t) = \dfrac{\partial^{m}}{\partial x^{m}}u(x,t)$	$W_k(x) = \dfrac{\partial^{m}}{\partial x^{m}}U_k(x)$

Remark. In Table1, Γ stands for Gamma function which is expressed as

$$\Gamma(z) := \int_0^{\infty} e^{-t}t^{z-1}dt, z \in \mathbb{C}. \tag{3.6}$$

It should be noted that the Gamma function is the continuous extension to the function. In the entire paper we will be employing the recursive relation $\Gamma(z+1) = z\Gamma(z), z > 0$ to compute the value of Gamma function of all real numbers by having idea only on the value of the Gamma function between 1 and 2.

Now, for illustration purpose we solve the Cahn–Hilliard equation in standard by using the RDTM

$$L\big(u(x,t)\big) + R(u(x,t)) + N(u(x,t)) = 0,$$ (3.7)

subject to the initial conditions

$$u(x,0) = f(x)$$ (3.8)

where $L = \dfrac{\partial}{\partial t}$ denotes a linear operator, $N\big(u(x,t)\big) = u^3$ has to do with the remaining linear term.

By applying the theorems in Table 1 above, we can construct the following recursive relation:

$$(k+1)U_k(x) = R\big(U(x,t)\big) - N\big(U_k(x)\big) + U_k(x)$$ (3.9)

where $R\big(U_k(x,t)\big)$, $U_k(x)$ and $N\big(U_k(x,t)\big)$ denote the transformations of $R\big(u(x,t)\big)$,

$u(x,t)$ and $N\big(u(x,t)\big)$ correspondingly. Now equation (3.8), which is the initial condition can be arranged as:

$$U_0(x) = f(x)$$ (3.10)

In order to obtain all other iterations, we initially substitute equation (3.10) into equation (3.9) and then we determine the values $U_k(x)$.

$$\tilde{u}(x,t) = \sum_{k=0}^{n} U_k(x)t^k$$ (3.11)

where n represents the number of iterations employ to obtain the approximate solution. Thus, the exact solution of the problem is stated as $u(x,t) = \lim_{n\to\infty} \tilde{u}(x,t)$.

4.   Application

Applying the RDTM to the one dimensional partial differential equation, we have the following relation

Example 1 Consider the following Korteureg-de – varies (generalized) partial differential equation [10, 24]

$$\partial_t u + \partial_x^3 u = \partial_x^5 u, \ x \in \mathbb{R}$$ (4.1)

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with initial condition

$$u_0(x,0) = \sin x \qquad (4.2)$$

where $u = u(x,t)$ is a function of the variables x and t.

Then, by applying the basic properties of the reduced differential transformation, we can obtain the transformed form of equation (4.1) as

$$U_{k+1}(x) = \frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha+\alpha+1)}\left(\frac{\partial^5}{\partial x^5}(U_k(x)) - \frac{\partial^3}{\partial x^3}(U_k(x))\right) \qquad (4.3)$$

where the t-dimensional spectrum function $U_k(x)$ is the transformed function.

From the initial condition (4.2) we write

$$U_0(x) = \sin x \qquad (4.4)$$

Substituting (4.4) into (4.3), we obtain the following $U_k(x)$ values successively

$$U_0 = sinx, U_1 = \frac{2cosx\Gamma(\alpha)}{\Gamma(2\alpha)}, U_2 = -\frac{4\Gamma(\alpha)\sin x}{\Gamma(3\alpha)}, U_3 = -\frac{8cosx\Gamma(\alpha)}{\Gamma(4\alpha)} ..... \qquad (4.5)$$

We continue in this manner and after a few iterations, the differential inverse transform of $\{U_k(x)\}_{k=0}^{n}$ will give the following approximate solution:

$$\tilde{u}(x,t) = \sum_{k=0}^{\infty} U_k(x)t^k \qquad (4.6)$$

$$= U_0(x) + U_1(x)t^\alpha + U_2(x)t^{2\alpha} + U_3(x)t^{3\alpha} + ... \qquad (4.7)$$

$$Sinx + \frac{2cosx\Gamma(\alpha)}{\Gamma(2\alpha)} - \frac{4\Gamma(\alpha)sinx}{\Gamma(3\alpha)} + \frac{16sinx\Gamma(\alpha)}{\Gamma(5\alpha)}... \qquad (4.8)$$

$$Sinx + \frac{2cosx\Gamma(\alpha)}{\Gamma(2\alpha)}t^\alpha - \frac{4\Gamma(\alpha)sinx}{\Gamma(3\alpha)}t^{2\alpha} - \frac{8cosx\Gamma(\alpha)}{\Gamma(4\alpha)}t^{3\alpha} + \frac{16sinx\Gamma(\alpha)}{\Gamma(5\alpha)}t^{4\alpha}... \qquad (4.9)$$

Hence, the approximate solution is obtained. Now, the numerical results of RTDM are computed for different

values of $\alpha = 0.6, \alpha = 0.8, \alpha = 0.9, \alpha = 1.0$ and different values of x and t. The numerical solution of RDTM is compared with RK4 method and graphs are subsequently shown.

Example 2. Consider the following the Korteureg-de-varies equation [10, 24]

$$\partial_t u + \partial_x^3 u + 2u \partial_x u - v \partial_x^2 u = 0, \ x \in \mathbb{R}$$

(4.10)

with initial condition

$$u_0(x,0) = \cos x$$

(4.11)

where $u = u(u,t)$ is a function of the variables x and t .

Then, by using the basic properties of the reduced differential transformation, we can find the transformed form of equation (4.7) as

$$U_{k+1}(x) = \frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha+\alpha+1)} \left(-2\sum_{r=0}^{k} U_r \frac{\partial}{\partial x} U_{k-r}(x) + v \frac{\partial^2}{\partial x^2}(U_k(x)) - \frac{\partial^3}{\partial x^3}(U_k(x)) \right)$$

(4.12)

where the t -dimensional spectrum function $U_k(x)$ are the transformed function.

From the initial condition (4.8) we write

$$U_0(x) = \sin x$$

(4.14)

Now, substituting (4.14) into (4.13), we obtain the following $U_k(x)$ values successively

Finally the differential inverse transform of $U_k(x)$ gives

$$\tilde{u}(x,t) = \sum_{k=0}^{\infty} U_k(x) t^k$$

(4.15)

$$= U_0(x) + U_1(x)t^\alpha + U_2(x)t^{2\alpha} + U_3(x)t^{3\alpha} + ...$$

(4.16)

$$U_0 = cosx, \quad U_1 = -\frac{\Gamma(\alpha)(cosx + sinx - sin2x)}{\Gamma(2\alpha)},$$

$$U_2 = -\frac{\Gamma(\alpha)(cosx - 10sin2x + 3cos3x - 2sinx + 6sin2x)}{\Gamma(3\alpha)},$$

$$U_3 = \frac{1}{4\Gamma(2\alpha)^2\Gamma(4\alpha)}\Gamma(\alpha)(\Gamma(\alpha)\Gamma(3\alpha)(-12 + 8cosx + 6cos2x - 16cos3x + 14cos4x - sinx + 8sin2x + 7sin3x + 2sin4x)$$

$$+\Gamma(2\alpha)^2(1 - 136cos3x + 260cos2x - 592cos3x + 99cos4x - 9sinx - 240sin2x + 259sin3x + 52sin4x))$$

$$Cosx - \frac{\Gamma(\alpha)(cosx + sinx - sin2x)}{\Gamma(2\alpha)}t^\alpha - \frac{\Gamma(\alpha)(cosx - 10sin2x + 3cos3x - 2sinx + 6sin2x)}{\Gamma(3\alpha)}t^{2\alpha} +$$

$$\frac{1}{4\Gamma(2\alpha)^2\Gamma(4\alpha)}\Gamma(\alpha)(\Gamma(\alpha)\Gamma(3\alpha)(-12 + 8cosx + 6cos2x - 16cos3x + 14cos4x - sinx + 8sin2x + 7sin3x + 2sin4x)$$

$$+\Gamma(2\alpha)^2(1 - 136cos3x + 260cos2x - 592cos3x + 99cos4x - 9sinx - 240sin2x + 259sin3x + 52sin4x))t^{3\alpha}$$

(4.17)

Thus, the approximate solution of equation (4.10) is obtained. Now, the numerical results of RTDM are also computed for different values of $\alpha = 0.6, \alpha = 0.8, \alpha = 0.9, \alpha = 1.0$ for different values of x and t. The numerical solution of RDTM is compared with RK4 method and graphs are subsequently shown.

5.   Results and Discussion

The fractional reduced differential transform has been employed to examine the dynamics of Korteureg-de – varies. The equations were solved using different values of α and Runge-Kutta fourth order method. The approximate solutions converged rapidly. Then we determine numerical results of the approximate solution for varying values of $\alpha = 0.5, \alpha = 0.7, \alpha = 1.0$ with same values of x and t. The results are graphically plotted and compared with varying values of α are presented in Figure 1 and Figure 2. All the computations were carried out in Mathematica 10.0 version.   From Figure 1 (a, c, e) as in equation 4.1, one can see that the values of the approximate solution of different grid points and different values of α determined by FRDTM are very related   to the values of the Runge-Kutta fourth order method in Figure1   (b, d, f) with high precision and the accuracy improves as the order of approximation goes up.
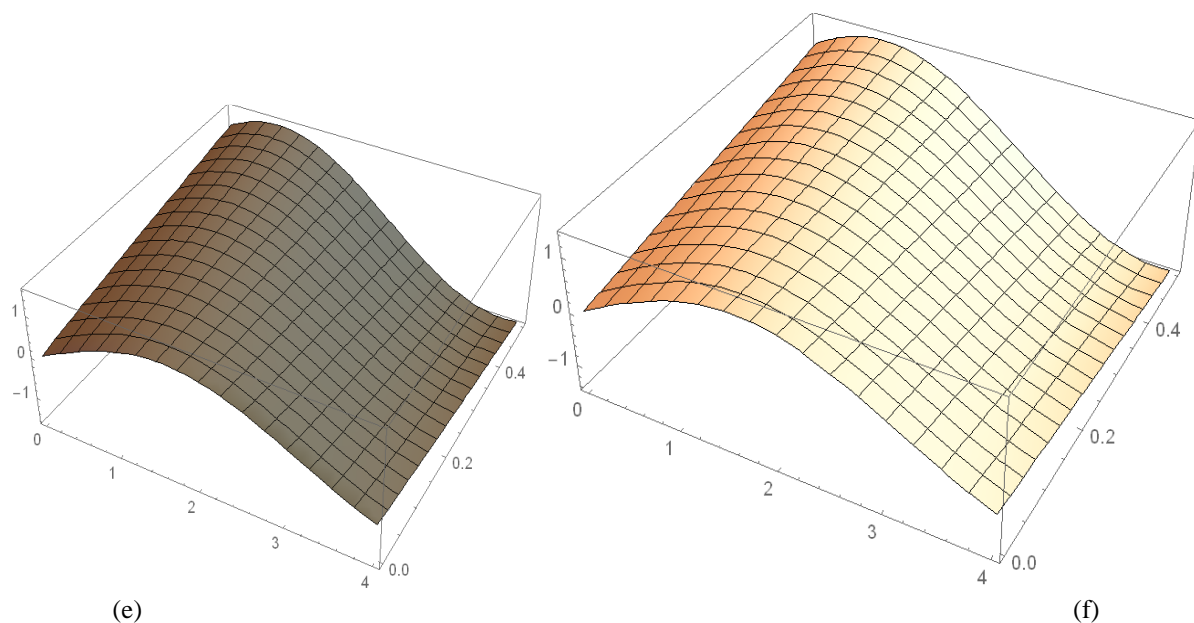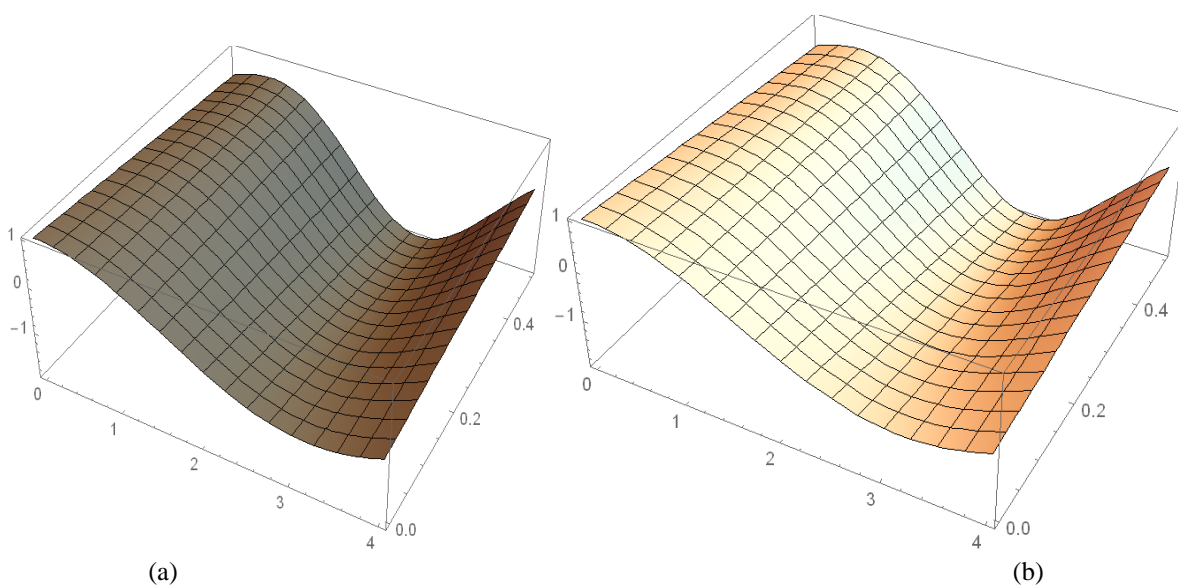
(a)

(b)

(c)

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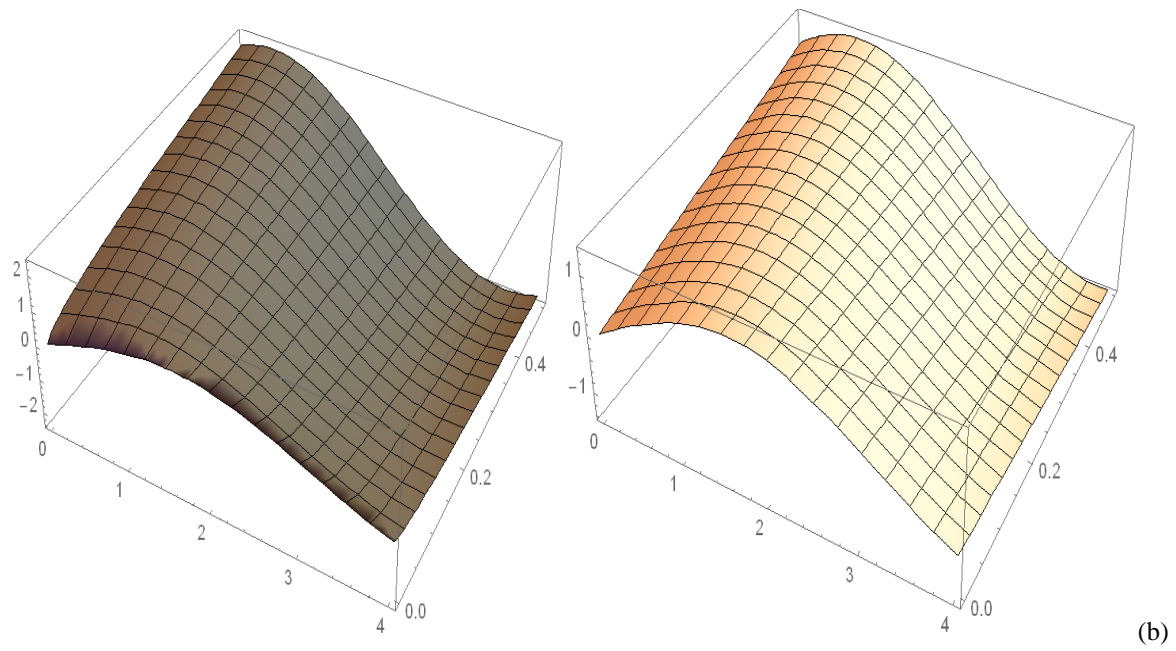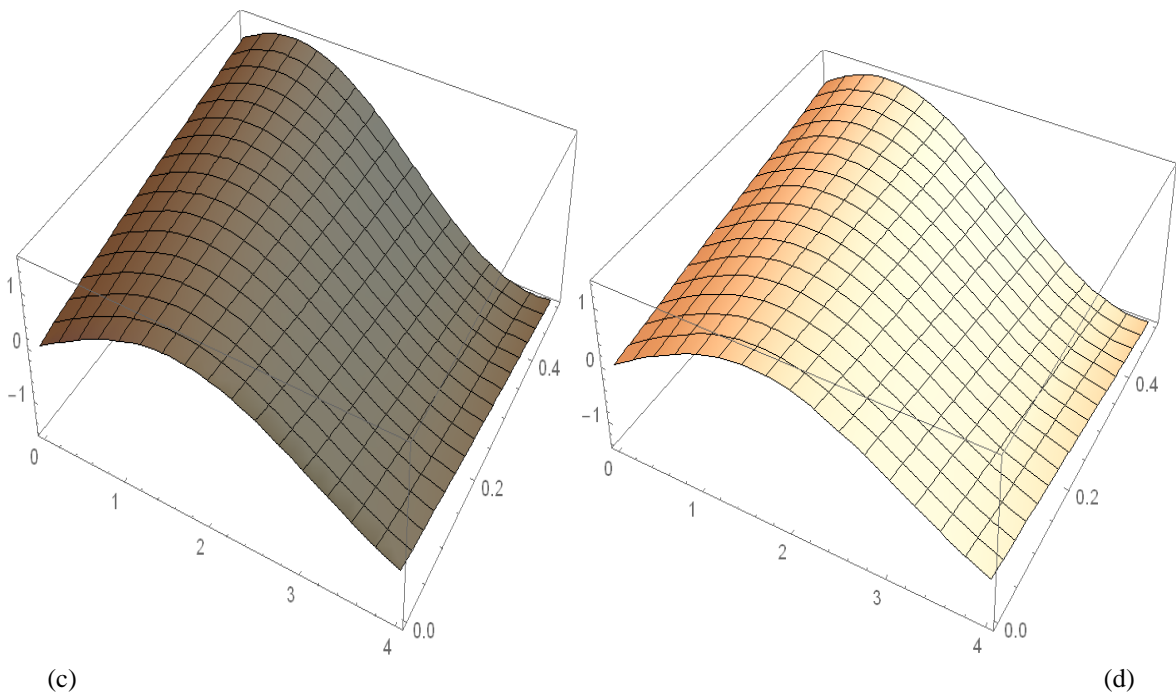
Figure 1: The approximate solution for equation 4.1 when $\alpha = 0.5, \alpha = 0.7, \alpha = 1.0$ (a, c, e) and Runge-Kutta 4th order (b, d, f)

Similarly, one can also observe same characteristics in Figure 2 (a, c, e) based on equation 4.2 that the values of the approximate solution of different grid points and different values of α which are determined by FRDTM matches with the values of the Runge-Kutta fourth order method in Figure2 (b, d, f) with high accuracy and the precision.
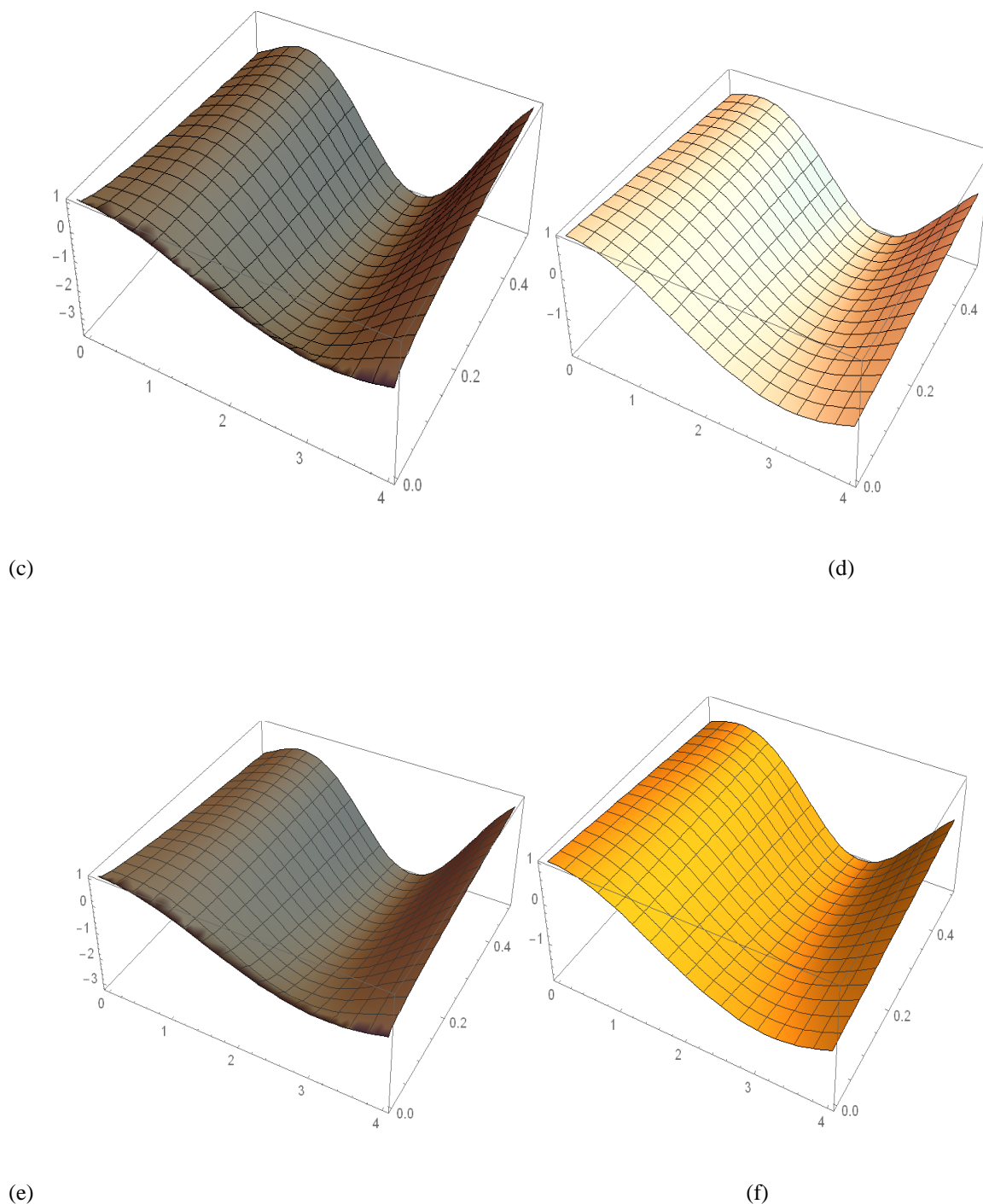

(a)

(b)

(c)　　　　　　　　　　　　　　　　　　　　　　　　　　　　　　(d)

(e)　　　　　　　　　　　　　　　　　　　　　　　　　　　　　　(f)

Figure 2: The approximate solution for equation 4.2 when $\alpha = 0.5, \alpha = 0.7, \alpha = 1.0$ (a, c, e) and Runge-Kutta 4$^{\text{th}}$ order (b, d, f)

6.　　Conclusion

In this present work, we have explored and demonstrated the power of FRDTM and apply to obtain approximate analytical solution of fractional Korteureg-de – varies for varying values of α and the results obtained in 4.1 and 4.2 are very close to the Rumge-Kutta 4$^{\text{th}}$ order results respectively. The method is employed directly without applying technique such linearization, transformation, discretization or even compelling

assumptions. The robustness of this method is shown from the computation results. The FRDTM presents a vital advancement in many fields over other current methods since it requires less computations as compare to other techniques available. In future we anticipate to apply this method to deal with other nonlinear fractional PDEs which are always characterized with many fields such as Biology, Environmental dynamics Mathematics and Engineering.

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