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General Closed Form Wave Solutions of Nonlinear Space-Time Fractional Differential Equation in Nonlinear Science

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Abstract

We have enucleated new and further exact general wave solutions, along with multiple exact traveling wave solutions of space-time nonlinear fractional Chan-Hillard equation, by applying a relatively renewed technique two variables $(G'/G, 1/G)$-expansion method. Also, based on fractional complex transformation and the properties of the modified Riemann-Liouville fractional order operator, the fractional partial differential equations are transforming into the form of ordinary differential equation. This method can be rumination of as the commutation of well-appointed (G'/G)-expansion method introduced by M. Wang and his colleagues In this paper, it is mentioned that the two variables $(G'/G, 1/G)$- expansion method is more legitimate, modest, sturdy and effective in the sense of theoretical and pragmatical point of view. Lastly, by treating computer symbolic program Mathematica, the uniqueness of our attained wave solutions are represented graphically and reveal a comparison in a submissive manner.

Keywords: $(G'/G, 1/G)$- expansion method; fractional Chan-Hillard equation; fractional derivative; traveling wave solution.

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1. Introduction

The study of fractional calculus is becoming an enormous important topic in the field of applied mathematics and mathematical analysis. The first appearance of the concept of a fractional calculus is found in 1695 [1], when Gottfried Wilhelm Leibniz suggested the possibility of fractional derivatives for the first time. As far as the existence of such a theory is perturbated, the foundations of the subject were placed by Liouville in a paper from 1832, and the fractional derivative of a power function apprehended by Riemann in 1847 [2]. Fractional calculus is the generalization of classical ordinary differentiation and integration and broadly depicts as a powerful tool for modeling complex systems, specifically for viscoelastic materials. In the last few decades, it is noticeable that the study of explicit exact solitary wave solutions of NFDEs plays a drastic role because of their prospective implementations in various scientific and technological fields, specifically in traffic flow, solid state physics, mathematical physics, mechanics, plasma physics, signal processing, bioengineering, optical fibers, geochemistry, stochastic dynamical systems, nonlinear optics, economics and business [3, 4]. For swift upward inclination of different symbolic computer programming tool, in this recent time many researchers and scholars have been attracted to solve nonlinear fractional differential equations by interposing several explicit, effective and powerful approaches. Researchers have been introduced different methods such as auxiliary equation method [5-7], the first integral method [8, 9], fractional $(\boldsymbol{G'/G})$-expansion method [10,11], fractional reduced differential transform method [12], Exp-function method [13,14], fractional sub-equation method [15-17], generalized kudryashov method [18-21], generalized $(\boldsymbol{G'/G})$-expansion method [22] and so on. In this present time, based on the original $(\boldsymbol{G'/G})$-expansion method different modified technique has been developed such as: generalized $(\boldsymbol{G'/G})$expansion, $(\boldsymbol{G'/G^2})$-expansion method, extended $(\boldsymbol{G'/G})$-expansion method etc. The key idea of the original single variable $(\boldsymbol{G'/G})$-expansion method is that, the closed form exact traveling wave solution of nonlinear PDEs can be exposed by a polynomial in one variable $(\boldsymbol{G'/G})$, in which $\boldsymbol{G=G(\eta)}$ conciliates the second order LODE $\boldsymbol{G''(\eta)+\lambda G'(\eta)+\mu G(\eta)=0}$, wherein $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are nonzero constant and $\boldsymbol{G'}$is the derivative of $\boldsymbol{G}$. This equation can also be treated as an auxiliary equation. However, inspired and motivated by the ongoing research in this arena, we also introduce and make best use of $(\boldsymbol{G'/G, 1/G})$-expansion method, which can be envisaged as the modification of the original $(\boldsymbol{G'/G})$-expansion method [23]. The main point of this two variables $(\boldsymbol{G'/G, 1/G})$-expansion method is that the exact traveling wave solution of nonlinear PDEs can be revealed by a polynomial in the two variables $(\boldsymbol{G'/G})$ and $\boldsymbol{1/G}$ in which $\boldsymbol{G=G(\eta)}$ satisfies the second order LODE $\boldsymbol{G''(\eta)+\lambda\, G(\eta)=\mu}$, wherein $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are non-zero constants. The degree of this polynomial can be discerned by introducing homogeneous balance principle between the highest order derivatives with highest order nonlinear term appearing in the conferred nonlinear PDEs. Also, the coefficient of this polynomial can be ascertained by solving a set of algebraic equations resulted from the process of using this technique. More recently, some researchers as like M. H. Uddin and his colleagues [24] expressed different types of new exact solitary wave solutions by dealing two variables $(\boldsymbol{G'/G, 1/G})$-expansion method. Further, E. Yasar and I. B. Giresunlu [25] acquired exact wave solutions of space-time fractional Chan-Allen and klein-Gordon equation by treating $(\boldsymbol{G'/G, 1/G})$-expansion method. Likewise, adopting similar technique, M. Topsakal and his colleagues [26] have performed three different types of exact solitary wave solutions for space-time fractional mBBM and modified nonlinear Kawahara equation. Activated by the ceaseless research in the analogous topics, for the first time we disclose new and further general exact traveling and solitary wave solution for the system of nonlinear space-time fractional Chan-Hillard equation by applying two variables $(G'/G, 1/G)$-expansion method. The principle motive for selecting this method, it gives the solutions in more general form. Moreover, the advantages of our proposed method over the original (G'/G)-expansion method is that the solutions treating the first method recapture the solutions treating the second one. So, we can say that two variables $(G'/G, 1/G)$-expansion method is an extension of the single variable (G'/G)-expansion method. In this article, we have constructed new traveling wave solutions including symmetrical Fibonacci function solutions, hyperbolic function solutions and rational solutions of the space-time fractional Cahn Hillihard equation, the form of the equation is representing by [27]

$$D_t^\alpha u - \gamma D_x^\alpha u - 6u(D_x^\alpha u)^2 - (3u^2 - 1)D_x^\alpha(D_x^\alpha u) + D_x^\alpha\left(D_x^\alpha\left(D_x^\alpha(D_x^\alpha u)\right)\right) = 0, t > 0, 0 < \alpha \leq 1 . \tag{1.1}$$

The rest of this ongoing paper is methodized as follows: In section 2, complying basic definitions along with the properties of the modified Riemann-Liouville fractional order derivative. In section 3, we propound the main theme of the renewed $(G'/G, 1/G)$- expansion method. In section 4, we implement this technique to find new exact solitary wave solutions of the space-time FDEs written above. The nature of the solutions along with their graphical representation is submitted in section 5. In section 6, we recount the results and discussion and lastly, in section 7, set out concluding remarks in a comprehensive manner.

2. Properties of the modified Riemann-Liouville fractional order derivative

There are several kinds of fractional derivative operators in fractional calculus. In this paper, we resort most suitable modified Riemann-Liouville derivative, which can be derived by Riemann for fractional derivative.

Definition 1: Consider that, the Jumarie's modified Riemann –Liouville derivative of order α with the continuous function$f: R \to R$, $x \to f(x)$ is stated by the expression [28]

$$D_x^\alpha f(x) = \begin{cases} \frac{1}{\Gamma(-\alpha)} \int_0^x (x-\eta)^{-\alpha-1}\big(f(\eta) - f(0)\big)\, d\eta, & \alpha < 0, \\ \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x-\eta)^{-\alpha}\big(f(\eta) - f(0)\big)\, d\eta, & 0 < \alpha < 1, \\ \left(f^{(n)}(x)\right)^{(\alpha-n)}, & n \leq \alpha \leq n+1, \ n \geq 1 \end{cases} \tag{2.1}$$

Where "Γ" is the Gamma function and is defined by

$$\Gamma(\alpha) = \lim_{n\to\infty} \frac{n!\, n^\alpha}{\alpha\,(\alpha+1)(\alpha+2)\ldots\ldots\ldots(\alpha+n)} \tag{2.2}$$

or $$\Gamma(x) = \int_0^\infty e^{-t}\, t^{(x-1)} dx. \tag{2.3}$$

Definition 2: The Mittag-Leffler function with two parameters is defined as:

$$E_{\alpha,\beta}^{(x)} = \sum_{i=0}^{\infty} \frac{x^i}{\Gamma(\alpha i+\beta)}, \qquad \Re e(\alpha) > 0, \beta, x \epsilon\, \mathbb{C}, \tag{2.4}$$

This function is utilized to solve FDEs as the exponential function in integer order systems. In the sake of derivative, some essential postulates which we use further in this paper are chosen as follows:

Postulate 1:

$$\boldsymbol{D_x^{\alpha}\, x^{\gamma} = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)} x^{\gamma-\alpha}}, \qquad \boldsymbol{\gamma > 0} \tag{2.5}$$

here $\boldsymbol{\gamma}$ is real number.

Postulate 2:

$$\boldsymbol{D_x^{\alpha}\big(C\, f(x)\big) = C D_x^{\alpha} f(x)}, \quad \boldsymbol{C = constant}. \tag{2.6}$$

Postulate 3:

$$\boldsymbol{D_x^{\alpha}\big(af(x) + bg(x)\big) = aD_x^{\alpha} f(x) + bD_x^{\alpha} g(x)}, \tag{2.7}$$

herein $\boldsymbol{a}$ and $\boldsymbol{b}$ are arbitrary constant.

Postulate 4:

$$\boldsymbol{D_x^{\alpha} f(\eta) = \frac{df}{d\eta}\, D_x^{\alpha}(\eta)}, \tag{2.8}$$

where $\boldsymbol{\eta = g(x)}$.

3. Main theme of $(\boldsymbol{G'/G, 1/G})$- expansion method

In this section, we explain the major conception of $(G'/G, 1/G)$- expansion method for obtaining the exact solitary wave solutions of the nonlinear time fractional differential equation. As per usual, we envisage the second order linear ordinary differential equation (LODE) in $G = G(\eta)$ as

$$G''(\eta) + \lambda G(\eta) = \mu, \tag{3.1}$$

and we assume two rational functions ϕ and ψ as

$$\phi = \frac{G'(\eta)}{G(\eta)}, \psi = \frac{1}{G(\eta)}, \tag{3.2}$$

here λ, μ are two arbitrary constant and $G'$is the derivative of G.

From eq. (3.1) and (3.2) yields

$$\phi' = -\phi^2 + \mu\psi - \lambda, \quad \psi' = -\phi\psi. \tag{3.3}$$

The solution of LODE (3.1), accomplish the following three distinct types:

Type 1: If $\lambda < 0$, the general solution of LODE (9) gives

$$G(\eta) = A_1 \sinh\left(\sqrt{-\lambda}\,\eta\right) + A_2 \cosh\left(\sqrt{-\lambda}\,\eta\right) + \frac{\mu}{\lambda}, \tag{3.4}$$

here A_1 and A_2 are two arbitrary constant. Therefore, from eqns. (3.2), (3.3) and (3.4) can be inferred the following relation

$$\psi^2 = \frac{-\lambda}{\lambda^2\varepsilon+\mu^2}(\phi^2 - 2\mu\psi + \lambda), \text{ where } \varepsilon = A_1^2 - A_2^2. \tag{3.5}$$

Type 2: If $\lambda > 0$, the general solution of LODE (3.1) is

$$G(\eta) = A_1 \sin\left(\sqrt{\lambda}\,\eta\right) + A_2 \cos\left(\sqrt{\lambda}\,\eta\right) + \frac{\mu}{\lambda}, \tag{3.6}$$

and similarly the above mentioned way, from eqns. (3.2), (3.3) and (3.6) the corresponding relations are

$$\psi^2 = \frac{\lambda}{\lambda^2\varepsilon-\mu^2}(\phi^2 - 2\mu\psi + \lambda), \text{ where } \varepsilon = A_1^2 + A_2^2. \tag{3.7}$$

Type 3: Finally, if $\lambda = 0$, the general solution of LODE (3.1) takes the form

$$G(\eta) = \frac{\mu}{2}\eta^2 + A_1\eta + A_2, \tag{3.8}$$

Consequently, we have

$$\psi^2 = \frac{1}{A_1^2 - 2\mu A_2}(\phi^2 - 2\mu\psi). \tag{3.9}$$

Now suppose we have a nonlinear fractional differential equation (FDE), say in two independent variables x and t is of the form

$$L(u, D_t^\alpha u, D_x^\alpha u, D_t^\alpha D_t^\alpha u, D_t^\alpha D_x^\alpha u, D_x^\alpha D_x^\alpha u, \dots\dots\dots) = 0, \tag{3.10}$$

where L is a polynomial in $u = u(x,t)$ and its various partial fractional derivatives. The main structure of the $(G'/G, 1/G)$- expansion method are present as the following stepwise:

First Step: Take into account, the traveling wave variable

$$u(x,t) = u(\eta), \eta = \frac{k\,x^\alpha}{\Gamma(1+\alpha)} + \frac{c\,t^\alpha}{\Gamma(1+\alpha)} + \eta_0, \tag{3.11}$$

where k, c and η_0 are non-zero arbitrary constant. Substituting eq. (3.11) and the values of $u = u(x,t)$ and its various fractional derivatives into eq. (3.10), it transform into the following ordinary differential equation

(ODE) in the form

$$T(u, u', u'', u''', \dots\dots) = 0, \tag{3.12}$$

here $'$ indicates $\frac{d}{d\eta}$ and so on. If needed, we integrate eq. (3.12) one or more times and assuming the constant of integration to be zero.

Second Step: In accordance with the above mentioned technique theorize that the exact solution of eq. (3.12) can be disclosed by a finite power series in two variables ϕ and ψ as follows

$$u(\eta) = \sum_{i=0}^{N} a_i \phi^i + \sum_{j=1}^{N} b_j \phi^{j-1} \psi, \tag{3.13}$$

where $a_i (i = 0,1,2, \dots\dots, N)$ and $b_j (j = 1,2, \dots\dots, N)$ are constants, which will be determined later. As for instance, evaluating the values of positive integer N by homogeneous balance principle, balancing the highest order derivative with the highest order nonlinear term appears in eq. (3.12). But all the times balance number N is not positive, sometimes they are in fraction or negative. In this case, we use the following transformation

(a) If $N = \frac{q}{p}$, where $\frac{q}{p}$ is a fraction in the lowest terms, then

$$u(\eta) = v^{\frac{q}{p}}(\eta), \tag{3.14}$$

then substitute eq. (3.14) into eq. (3.12) to attain a equation in the renewed function $u(\eta)$ with a positive integer balance number.

(b) If N is a negative number, then

$$u(\eta) = v^N(\eta), \tag{3.15}$$

and set eq. (3.15) into eq. (3.12) to obtain a equation in the new function $u(\eta)$ with a positive integer balance number.

Third Step: Substituting eq. (3.13) into eq. (3.12), also operating eqns. (3.3) and (3.5) will transform into a polynomial in ϕ and ψ, in which the degree of ψ is not longer than 1. Equalizing all of the coefficient of this polynomial to zero, yields a system of algebraic equations for $a_i (i = 0,1,2, \dots\dots, N)$, $b_j (j = 1,2, \dots\dots, N)$, k, c, μ, λ $(\lambda < 0)$, $A_1$and A_2.

Fourth Step: Solve the system which is obtained in third step with the help of any computer symbolic program, like Mathematica and substituting the values of a_i, b_j, k, c, μ, λ $(\lambda < 0)$, $A_1$and A_2 into eq. (3.13) and finally, we come up with different types of exact wave solutions of eq. (3.10) represented by the hyperbolic functions.

Fifth Step: Likewise, pursuing the third step and fourth step, plugging eq. (3.13) into eq. (3.12), treating eq.

(3.3) and eq. (3.7) (or eq. (3.3) and eq. (3.9)), we come up with the solution of eq. (3.10) in case of trigonometric functions (or by rational functions) as proceeding before.

4. Formulation of the solutions

In this section, the $(\boldsymbol{G'/G, 1/G})$- expansion method has been applied to visualizing different new exact wave solutions of the nonlinear space-time fractional equation namely Cahn-Hillihard equation (1.1).

4.1 Space-time fractional Cahn-Hillihard equation

In this subsection, we will make best use of the $(G'/G, 1/G)$- expansion method to constitute the exact traveling and solitary wave solutions of the space-time fractional Cahn-Hillihard equation (1.1). Take into account, the following wave transformation

$$u(x,t) = u(\eta), \eta = \frac{k\,x^{\alpha}}{\Gamma(1+\alpha)} + \frac{c\,t^{\alpha}}{\Gamma(1+\alpha)} + \eta_0, \tag{4.1.1}$$

Where k, c and η_0 are nonzero constants. This wave transformation consents us eq. (1.1) reducing into an ODE

$$(c - k\,\gamma)u' - 6k^2u(u')^2 - 3k^2u^2u'' + k^2u'' + k^4u^{(4)} = 0, \tag{4.1.2}$$

here prime denotes the derivative with respect to η

Balancing the number is $N = 1$, can be acquired by balancing between the highest order derivative $u^{(4)}$ and the highest order nonlinear term u^2u''. On putting the value of $N = 1$ in eq. (3.13), then the solution formula becomes

$$u(\eta) = a_0 + a_1\phi + b_1\psi, \tag{4.1.3}$$

where a_0, a_1, $b_1$are constants to be discerned later. Therefore, the above mentioned three annotations are studied as follows

Type 1: When $\lambda < 0$ (Hyperbolic function solutions)

Plugging the value of $u(\eta)$ from eq. (4.1.3), into eq. (4.1.2), in addition with eq. (3.3) and (3.5), the left hand side of eq. (4.1.2) becomes a polynomial in ϕ and ψ. Placing each coefficient of the polynomial equal to zero, we arrives a system of algebraic equations as below

$$\phi: 2k^2\lambda a_1 + 16k^4\lambda^2 a_1 - \frac{30k^4\lambda^2\mu^2 a_1}{\mu^2+\lambda^2\sigma} - 6k^2\lambda a_0^2 a_1 - 6k^2\lambda^2 a_1^3 + \frac{6k^2\lambda^2\mu^2 a_1^3}{\mu^2+\lambda^2\sigma} -$$
$$\frac{36k^2\lambda^2\mu a_0 a_1 b_1}{\mu^2+\lambda^2\sigma} - \frac{54k^2\lambda^3\mu^2 a_1 b_1^2}{(\mu^2+\lambda^2\sigma)^2} + \frac{24k^2\lambda^3 a_1 b_1^2}{\mu^2+\lambda^2\sigma} = 0,$$

$$\phi^2: -ca_1 + k\gamma a_1 - 24k^2\lambda a_0 a_1^2 + \frac{6k^2\lambda\mu^2 a_0 a_1^2}{\mu^2+\lambda^2\sigma} - \frac{12k^4\lambda^2\mu^3 b_1}{(\mu^2+\lambda^2\sigma)^2} + \frac{k^2\lambda\mu b_1}{\mu^2+\lambda^2\sigma} + \frac{47k^4\lambda^2\mu b_1}{\mu^2+\lambda^2\sigma} -$$

$\frac{3k^2\lambda\mu a_0^2 b_1}{\mu^2+\lambda^2\sigma} + \frac{12k^2\lambda^2\mu^3 a_1^2 b_1}{(\mu^2+\lambda^2\sigma)^2} - \frac{57k^2\lambda^2\mu a_1^2 b_1}{\mu^2+\lambda^2\sigma} - \frac{12k^2\lambda^2\mu^2 a_0 b_1^2}{(\mu^2+\lambda^2\sigma)^2} + \frac{24k^2\lambda^2 a_0 b_1^2}{\mu^2+\lambda^2\sigma} - \frac{12k^2\lambda^3\mu^3 b_1^3}{(\mu^2+\lambda^2\sigma)^3} +$
$\frac{36k^2\lambda^3\mu b_1^3}{(\mu^2+\lambda^2\sigma)^2} = 0,$

ϕ^3: $2k^2a_1 + 40k^4\lambda a_1 - \frac{30k^4\lambda\mu^2 a_1}{\mu^2+\lambda^2\sigma} - 6k^2a_0^2a_1 - 18k^2\lambda a_1^3 + \frac{6k^2\lambda\mu^2 a_1^3}{\mu^2+\lambda^2\sigma} - \frac{36k^2\lambda\mu a_0 a_1 b_1}{\mu^2+\lambda^2\sigma} -$
$\frac{54k^2\lambda^2\mu^2 a_1 b_1^2}{(\mu^2+\lambda^2\sigma)^2} + \frac{60k^2\lambda^2 a_1 b_1^2}{\mu^2+\lambda^2\sigma} = 0,$

ϕ^4: $-18k^2a_0a_1^2 + \frac{36k^4\lambda\mu b_1}{\mu^2+\lambda^2\sigma} - \frac{45k^2\lambda\mu a_1^2 b_1}{\mu^2+\lambda^2\sigma} + \frac{18k^2\lambda a_0 b_1^2}{\mu^2+\lambda^2\sigma} + \frac{27k^2\lambda^2\mu b_1^3}{(\mu^2+\lambda^2\sigma)^2} = 0,$

ϕ^5: $24k^4a_1 - 12k^2a_1^3 + \frac{36k^2\lambda a_1 b_1^2}{\mu^2+\lambda^2\sigma} = 0,$

ψ: $c\mu a_1 - k\gamma\mu a_1 + 12k^2\lambda\mu a_0 a_1^2 - \frac{12k^2\lambda\mu^3 a_0 a_1^2}{\mu^2+\lambda^2\sigma} + k^2\lambda b_1 + 5k^4\lambda^2 b_1 + \frac{24k^4\lambda^2\mu^4 b_1}{(\mu^2+\lambda^2\sigma)^2} -$
$\frac{2k^2\lambda\mu^2 b_1}{\mu^2+\lambda^2\sigma} - \frac{28k^4\lambda^2\mu^2 b_1}{\mu^2+\lambda^2\sigma} - 3k^2\lambda a_0^2 b_1 + \frac{6k^2\lambda\mu^2 a_0^2 b_1}{\mu^2+\lambda^2\sigma} - 6k^2\lambda^2 a_1^2 b_1 - \frac{24k^2\lambda^2\mu^4 a_1^2 b_1}{(\mu^2+\lambda^2\sigma)^2} +$
$\frac{30k^2\lambda^2\mu^2 a_1^2 b_1}{\mu^2+\lambda^2\sigma} + \frac{24k^2\lambda^2\mu^3 a_0 b_1^2}{(\mu^2+\lambda^2\sigma)^2} - \frac{18k^2\lambda^2\mu a_0 b_1^2}{\mu^2+\lambda^2\sigma} + \frac{24k^2\lambda^3\mu^4 b_1^3}{(\mu^2+\lambda^2\sigma)^3} - \frac{24k^2\lambda^3\mu^2 b_1^3}{(\mu^2+\lambda^2\sigma)^2} + \frac{3k^2\lambda^3 b_1^3}{\mu^2+\lambda^2\sigma} = 0,$

$\phi\psi$: $-3k^2\mu a_1 - 45k^4\lambda\mu a_1 + \frac{60k^4\lambda\mu^3 a_1}{\mu^2+\lambda^2\sigma} + 9k^2\mu a_0^2 a_1 + 12k^2\lambda\mu a_1^3 - \frac{12k^2\lambda\mu^3 a_1^3}{\mu^2+\lambda^2\sigma} - cb_1 + k\gamma b_1 -$
$30k^2\lambda a_0 a_1 b_1 + \frac{72k^2\lambda\mu^2 a_0 a_1 b_1}{\mu^2+\lambda^2\sigma} + \frac{108k^2\lambda^2\mu^3 a_1 b_1^2}{(\mu^2+\lambda^2\sigma)^2} - \frac{75k^2\lambda^2\mu a_1 b_1^2}{\mu^2+\lambda^2\sigma} = 0,$

$\phi^2\psi$: $30k^2\mu a_0 a_1^2 + 2k^2b_1 + 28k^4\lambda b_1 - \frac{78k^4\lambda\mu^2 b_1}{\mu^2+\lambda^2\sigma} - 6k^2a_0^2b_1 - 39k^2\lambda a_1^2 b_1 +$
$\frac{96k^2\lambda\mu^2 a_1^2 b_1}{\mu^2+\lambda^2\sigma} - \frac{42k^2\lambda\mu a_0 b_1^2}{\mu^2+\lambda^2\sigma} - \frac{60k^2\lambda^2\mu^2 b_1^3}{(\mu^2+\lambda^2\sigma)^2} + \frac{15k^2\lambda^2 b_1^3}{\mu^2+\lambda^2\sigma} = 0,$

$\phi^3\psi$: $-60k^4\mu a_1 + 21k^2\mu a_1^3 - 36k^2a_0a_1b_1 - \frac{99k^2\lambda\mu a_1 b_1^2}{\mu^2+\lambda^2\sigma} = 0,$

$\phi^4\psi$: $24k^4b_1 - 36k^2a_1^2b_1 + \frac{12k^2\lambda b_1^3}{\mu^2+\lambda^2\sigma} = 0,$

$Const.$: $-c\lambda a_1 + k\gamma\lambda a_1 - 6k^2\lambda^2 a_0 a_1^2 + \frac{6k^2\lambda^2\mu^2 a_0 a_1^2}{\mu^2+\lambda^2\sigma} - \frac{12k^4\lambda^3\mu^3 b_1}{(\mu^2+\lambda^2\sigma)^2} + \frac{k^2\lambda^2\mu b_1}{\mu^2+\lambda^2\sigma} + \frac{11k^4\lambda^3\mu b_1}{\mu^2+\lambda^2\sigma} -$
$\frac{3k^2\lambda^2\mu a_0^2 b_1}{\mu^2+\lambda^2\sigma} + \frac{12k^2\lambda^3\mu^3 a_1^2 b_1}{(\mu^2+\lambda^2\sigma)^2} - \frac{12k^2\lambda^3\mu a_1^2 b_1}{\mu^2+\lambda^2\sigma} - \frac{12k^2\lambda^3\mu^2 a_0 b_1^2}{(\mu^2+\lambda^2\sigma)^2} + \frac{6k^2\lambda^3 a_0 b_1^2}{\mu^2+\lambda^2\sigma} - \frac{12k^2\lambda^4\mu^3 b_1^3}{(\mu^2+\lambda^2\sigma)^3} +$
$\frac{9k^2\lambda^4\mu b_1^3}{(\mu^2+\lambda^2\sigma)^2} = 0,$

On solving the above algebraic equations with the aid of symbolic software program, like Mathematica, we comes up with the following various sets of solutions

Family 1:

$$a_0 = 0, a_1 = -\frac{i}{\sqrt{\lambda}}, b_1 = -\frac{\sqrt{\mu^2+\lambda^2\sigma}}{\lambda}, k = -\frac{i\sqrt{2}}{\sqrt{\lambda}}, c = -\frac{i\sqrt{2}\gamma}{\sqrt{\lambda}} \tag{4.1.4}$$

Setting these above values into the equipped eq. (4.1.3), we attain the exact wave solution of eq. (1.1) as

$$u(\eta) = -\frac{\sqrt{\mu^2+\lambda^2\sigma}}{\lambda\left(A_1 sinh(\eta\sqrt{-\lambda})+A_2 cosh(\eta\sqrt{-\lambda})+\frac{\mu}{\lambda}\right)} - \frac{i\sqrt{-\lambda}\left(A_1 cosh(\eta\sqrt{-\lambda})+A_2 sinh(\eta\sqrt{-\lambda})\right)}{\sqrt{\lambda}\left(A_1 sinh(\eta\sqrt{-\lambda})+A_2 cosh(\eta\sqrt{-\lambda})+\frac{\mu}{\lambda}\right)}, \tag{4.1.5}$$

Where $\sigma = A_1^2 - A_2^2$ and $\eta = \left(-i\sqrt{\frac{2}{\lambda}}\left(\frac{x^\alpha}{\Gamma(1+\alpha)} + \frac{\gamma t^\alpha}{\Gamma(1+\alpha)}\right) + \eta_0\right)$.

Particularly, if we put $A_1 = 0$, $\mu = 0$ and $A_2 \neq 0$ in eq. (4.1.5), the kink solution is

$$u(\eta) = tanh\left(\left(-i\sqrt{\frac{2}{\lambda}}\left(\frac{x^\alpha}{\Gamma(1+\alpha)} + \frac{\gamma t^\alpha}{\Gamma(1+\alpha)}\right) + \eta_0\right)\sqrt{-\lambda}\right) - \sqrt{\sigma}\, sech\left(\left(-i\sqrt{\frac{2}{\lambda}}\left(\frac{x^\alpha}{\Gamma(1+\alpha)} + \frac{\gamma t^\alpha}{\Gamma(1+\alpha)}\right) + \eta_0\right)\sqrt{-\lambda}\right). \tag{4.1.6}$$

but if we take $A_2 = 0$, $\mu = 0$ and $A_1 \neq 0$ then the solution takes the form

$$u(\eta) = coth\left(\left(-i\sqrt{\frac{2}{\lambda}}\left(\frac{x^\alpha}{\Gamma(1+\alpha)} + \frac{\gamma t^\alpha}{\Gamma(1+\alpha)}\right) + \eta_0\right)\sqrt{-\lambda}\right) - \sqrt{\sigma}\, csch\left(\left(-i\sqrt{\frac{2}{\lambda}}\left(\frac{x^\alpha}{\Gamma(1+\alpha)} + \frac{\gamma t^\alpha}{\Gamma(1+\alpha)}\right) + \eta_0\right)\sqrt{-\lambda}\right). \tag{4.1.7}$$

Family 2:

$$a_0 = 0,\ a_1 = \frac{i}{\sqrt{\lambda}}, b_1 = -\frac{\sqrt{\mu^2+\lambda^2\sigma}}{\lambda}, k = \frac{i\sqrt{2}}{\sqrt{\lambda}} \text{ and } c = \frac{i\sqrt{2}\gamma}{\sqrt{\lambda}}, \tag{4.1.8}$$

Substituting these values into the arranged eq. (4.1.3), the exact solution of eq. (1.1) is

$$u(\eta) = -\frac{\sqrt{\mu^2+\lambda^2\sigma}}{\lambda\left(A_1 sinh(\eta\sqrt{-\lambda})+A_2 cosh(\eta\sqrt{-\lambda})+\frac{\mu}{\lambda}\right)} + \frac{i\sqrt{-\lambda}\left(A_1 cosh(\eta\sqrt{-\lambda})+A_2 sinh(\eta\sqrt{-\lambda})\right)}{\sqrt{\lambda}\left(A_1 sinh(\eta\sqrt{-\lambda})+A_2 cosh(\eta\sqrt{-\lambda})+\frac{\mu}{\lambda}\right)}, \tag{4.1.9}$$

Where $\sigma = A_1^2 - A_2^2$ and $\eta = \left(i\sqrt{\frac{2}{\lambda}}\left(\frac{x^\alpha}{\Gamma(1+\alpha)} + \frac{\gamma t^\alpha}{\Gamma(1+\alpha)}\right) + \eta_0\right)$.

For auspicious values, put $A_1 = 0$, $\mu = 0$ and $A_2 \neq 0$ in eq. (4.1.9), the solution obtains

$$u(\eta) = -\left(tanh\left(\left(i\sqrt{\frac{2}{\lambda}}\left(\frac{x^\alpha}{\Gamma(1+\alpha)} + \frac{\gamma t^\alpha}{\Gamma(1+\alpha)}\right) + \eta_0\right)\sqrt{-\lambda}\right) + \sqrt{\sigma}\, sech\left(\left(i\sqrt{\frac{2}{\lambda}}\left(\frac{x^\alpha}{\Gamma(1+\alpha)} + \frac{\gamma t^\alpha}{\Gamma(1+\alpha)}\right) + \eta_0\right)\sqrt{-\lambda}\right)\right). \tag{4.1.10}$$

Also if we set $A_2 = 0$, $\mu = 0$ and $A_1 \neq 0$, we attain

$$u(\eta) = -\left(coth\left(\left(i\sqrt{\frac{2}{\lambda}}\left(\frac{x^\alpha}{\Gamma(1+\alpha)} + \frac{\gamma t^\alpha}{\Gamma(1+\alpha)}\right) + \eta_0\right)\sqrt{-\lambda}\right) + \sqrt{\sigma}\, csch\left(\left(i\sqrt{\frac{2}{\lambda}}\left(\frac{x^\alpha}{\Gamma(1+\alpha)} + \frac{\gamma t^\alpha}{\Gamma(1+\alpha)}\right) + \eta_0\right)\sqrt{-\lambda}\right)\right). \tag{4.1.11}$$

Family 3:

$$a_0 = \frac{\frac{\mu^3}{\sqrt{\mu^2-2\lambda^2\sigma}} + \frac{\lambda^2\mu\sigma}{\sqrt{\mu^2-2\lambda^2\sigma}}}{\mu^2+\lambda^2\sigma},\ a_1 = 0,\ b_1 = -\frac{2(\mu^2+\lambda^2\sigma)}{\lambda\sqrt{\mu^2-2\lambda^2\sigma}},\ k = -\frac{\sqrt{2}\sqrt{\mu^2+\lambda^2\sigma}}{\sqrt{-\lambda\mu^2+2\lambda^3\sigma}},\ c = -\frac{\sqrt{2}\gamma\sqrt{\mu^2+\lambda^2\sigma}}{\sqrt{-\lambda(\mu^2-2\lambda^2\sigma)}}, \tag{4.1.12}$$

By using these above values into the eq. (4.1.3), we derive the exact traveling wave solution of eq. (1.1) is

$$u(\eta) = \frac{\frac{\mu^3}{\sqrt{\mu^2-2\lambda^2\sigma}} + \frac{\lambda^2\mu\sigma}{\sqrt{\mu^2-2\lambda^2\sigma}}}{\mu^2+\lambda^2\sigma} - \frac{2(\mu^2+\lambda^2\sigma)}{\lambda\sqrt{\mu^2-2\lambda^2\sigma}\left(A_1 sinh(\eta\sqrt{-\lambda}\,) + A_2 cosh(\eta\sqrt{-\lambda}\,) + \frac{\mu}{\lambda}\right)}, \tag{4.1.13}$$

herein $\sigma = A_1^2 - A_2^2$ and $\eta = \left(-\sqrt{\frac{2(\mu^2+\lambda^2\sigma)}{-\lambda(\mu^2-2\lambda^2\sigma)}}\left(\frac{x^\alpha}{\Gamma(1+\alpha)} + \frac{\gamma t^\alpha}{\Gamma(1+\alpha)}\right) + \eta_0\right)$.

For special values, take $A_1 = 0$, $\mu = 0$ and $A_2 \neq 0$, the solution comes up

$$u(\eta) = i\sqrt{2\sigma} sech\left(\left(-\sqrt{\frac{2(\mu^2+\lambda^2\sigma)}{-\lambda(\mu^2-2\lambda^2\sigma)}}\left(\frac{x^\alpha}{\Gamma(1+\alpha)} + \frac{\gamma t^\alpha}{\Gamma(1+\alpha)}\right) + \eta_0\right)\sqrt{-\lambda}\right). \tag{4.1.14}$$

while, we set $A_2 = 0$, $\mu = 0$ and $A_1 \neq 0$ then the solution transform into

$$u(\eta) = i\sqrt{2\sigma} csch\left(\left(-\sqrt{\frac{2(\mu^2+\lambda^2\sigma)}{-\lambda(\mu^2-2\lambda^2\sigma)}}\left(\frac{x^\alpha}{\Gamma(1+\alpha)} + \frac{\gamma t^\alpha}{\Gamma(1+\alpha)}\right) + \eta_0\right)\sqrt{-\lambda}\right). \tag{4.1.15}$$

Type 2: When $\lambda > 0$ (Trigonometric function solutions)

In the similar manner, as mentioned above in type 1, putting the values of $u(\eta)$ from eq. (4.1.3), into eq. (4.1.2), along with eq. (3.3) and (3.7), the left hand side of eq. (4.1.2) becomes a polynomial in ϕ and ψ. Taking each coefficient of the polynomial equal to zero, obtain a system of algebraic equations (for the sake of simplicity, the equations are not submitted here) for a_0, a_1, b_1, μ, λ and σ. Solving these system of equations and gaining the following sets of results

Family 4:

$$a_0 = 0,\ a_1 = \frac{i}{\sqrt{\lambda}},\ b_1 = \frac{\sqrt{\mu^2-\lambda^2\sigma}}{\lambda},\ k = -\frac{i\sqrt{2}}{\sqrt{\lambda}} \text{ and } c = -\frac{i\sqrt{2}\gamma}{\sqrt{\lambda}}, \tag{4.1.16}$$

Therefore, in this result the exact solution of eq. (1.1) derives

$$u(\eta)=\frac{\sqrt{\mu^2-\lambda^2\sigma}}{\lambda\left(A_1 sin(\eta\sqrt{\lambda})+A_2 cos(\eta\sqrt{\lambda})+\frac{\mu}{\lambda}\right)}+\frac{i\left(A_1 cos(\eta\sqrt{\lambda})-A_2 sin(\eta\sqrt{\lambda})\right)}{\left(A_1 sin(\eta\sqrt{\lambda})+A_2 cos(\eta\sqrt{\lambda})+\frac{\mu}{\lambda}\right)}, \tag{4.1.17}$$

here $\sigma=A_1^2+A_2^2$ and $\eta=\left(-\frac{i\sqrt{2}}{\sqrt{\lambda}}\left(\frac{x^\alpha}{\Gamma(1+\alpha)}+\frac{\gamma t^\alpha}{\Gamma(1+\alpha)}\right)+\eta_0\right)$.

Especially, by setting $A_1=0$, $\mu=0$ and $A_2\neq 0$, the periodic solution becomes

$$u(\eta)=i\left(\sqrt{\sigma}\, sec\left(\left(-\frac{i\sqrt{2}}{\sqrt{\lambda}}\left(\frac{x^\alpha}{\Gamma(1+\alpha)}+\frac{\gamma t^\alpha}{\Gamma(1+\alpha)}\right)+\eta_0\right)\sqrt{\lambda}\right)-\ tan\left(\left(-\frac{i\sqrt{2}}{\sqrt{\lambda}}\left(\frac{x^\alpha}{\Gamma(1+\alpha)}+\frac{\gamma t^\alpha}{\Gamma(1+\alpha)}\right)+\eta_0\right)\sqrt{\lambda}\right)\right). \tag{4.1.18}$$

Besides, for $A_2=0$, $\mu=0$ and $A_1\neq 0$, we have

$$u(\eta)=i\left(cot\left(\left(-\frac{i\sqrt{2}}{\sqrt{\lambda}}\left(\frac{x^\alpha}{\Gamma(1+\alpha)}+\frac{\gamma t^\alpha}{\Gamma(1+\alpha)}\right)+\eta_0\right)\sqrt{\lambda}\right)+\sqrt{\sigma}csc\left(\left(-\frac{i\sqrt{2}}{\sqrt{\lambda}}\left(\frac{x^\alpha}{\Gamma(1+\alpha)}+\frac{\gamma t^\alpha}{\Gamma(1+\alpha)}\right)+\eta_0\right)\sqrt{\lambda}\right)\right). \tag{4.1.19}$$

Family 5:

$$a_0=0,\ a_1=-\frac{i}{\sqrt{\lambda}},\ b_1=-\frac{\sqrt{\mu^2-\lambda^2\sigma}}{\lambda},\ k=\frac{i\sqrt{2}}{\sqrt{\lambda}} \text{ and } c=\frac{i\sqrt{2}\gamma}{\sqrt{\lambda}}, \tag{4.1.20}$$

Hence, closed form of exact traveling wave solution of eq. (1.1) is

$$u(\eta)=-\frac{\sqrt{\mu^2-\lambda^2\sigma}}{\lambda\left(A_1 sin(\eta\sqrt{\lambda})+A_2 cos(\eta\sqrt{\lambda})+\frac{\mu}{\lambda}\right)}-\frac{i\left(A_1 cos(\eta\sqrt{\lambda})-A_2 sin(\eta\sqrt{\lambda})\right)}{\left(A_1 sin(\eta\sqrt{\lambda})+A_2 cos(\eta\sqrt{\lambda})+\frac{\mu}{\lambda}\right)} \tag{4.1.21}$$

wherein $\sigma=A_1^2+A_2^2$ and $\eta=\left(\frac{i\sqrt{2}}{\sqrt{\lambda}}\left(\frac{x^\alpha}{\Gamma(1+\alpha)}+\frac{\gamma t^\alpha}{\Gamma(1+\alpha)}\right)+\eta_0\right)$.

In particular, if we input $A_1=0$, $\mu=0$ and $A_2\neq 0$, we achieve the solution is of the form

$$u(\eta)=i\left(\ tan\left(\left(\frac{i\sqrt{2}}{\sqrt{\lambda}}\left(\frac{x^\alpha}{\Gamma(1+\alpha)}+\frac{\gamma t^\alpha}{\Gamma(1+\alpha)}\right)+\eta_0\right)\sqrt{\lambda}\right)-\sqrt{\sigma}sec\left(\left(\frac{i\sqrt{2}}{\sqrt{\lambda}}\left(\frac{x^\alpha}{\Gamma(1+\alpha)}+\frac{\gamma t^\alpha}{\Gamma(1+\alpha)}\right)+\eta_0\right)\sqrt{\lambda}\right)\right). \tag{4.1.22}$$

Also, for $A_2=0$, $\mu=0$ and $A_1\neq 0$, yields

$$u(\eta)=-i\left(\ cot\left(\left(\frac{i\sqrt{2}}{\sqrt{\lambda}}\left(\frac{x^\alpha}{\Gamma(1+\alpha)}+\frac{\gamma t^\alpha}{\Gamma(1+\alpha)}\right)+\eta_0\right)\sqrt{\lambda}\right)+\sqrt{\sigma}csc\left(\left(\frac{i\sqrt{2}}{\sqrt{\lambda}}\left(\frac{x^\alpha}{\Gamma(1+\alpha)}+\frac{\gamma t^\alpha}{\Gamma(1+\alpha)}\right)+\eta_0\right)\sqrt{\lambda}\right)\right). \tag{4.1.23}$$

Family 6:

$$a_0=\frac{-\frac{\mu^3}{\sqrt{\mu^2+2\lambda^2\sigma}}+\frac{\lambda^2\mu\sigma}{\sqrt{\mu^2+2\lambda^2\sigma}}}{-\mu^2+\lambda^2\sigma},\ a_1=0,\ b_1=-\frac{2(\mu^2-\lambda^2\sigma)}{\lambda\sqrt{\mu^2+2\lambda^2\sigma}},\ k=-\frac{\sqrt{2}\sqrt{-\mu^2+\lambda^2\sigma}}{\sqrt{\lambda\mu^2+2\lambda^3\sigma}},\ c=-\frac{\sqrt{2}\gamma\sqrt{-\mu^2+\lambda^2\sigma}}{\sqrt{\lambda(\mu^2+2\lambda^2\sigma)}}, \tag{4.1.24}$$

Hence setting these values in eq. (4.1.3), the solution of eq. (1.1) becomes

$$u(\eta)=\frac{-\frac{\mu^3}{\sqrt{\mu^2+2\lambda^2\sigma}}+\frac{\lambda^2\mu\sigma}{\sqrt{\mu^2+2\lambda^2\sigma}}}{-\mu^2+\lambda^2\sigma}-\frac{2(\mu^2-\lambda^2\sigma)}{\lambda\sqrt{\mu^2+2\lambda^2\sigma}\left(A_1 sin(\eta\sqrt{\lambda})+A_2 cos(\eta\sqrt{\lambda})+\frac{\mu}{\lambda}\right)}, \quad (4.1.25)$$

herein $\sigma=A_1^2+A_2^2$ and $\eta=\left(-\sqrt{\frac{2(-\mu^2+\lambda^2\sigma)}{\lambda(\mu^2+2\lambda^2\sigma)}}\left(\frac{x^\alpha}{\Gamma(1+\alpha)}+\frac{\gamma t^\alpha}{\Gamma(1+\alpha)}\right)+\eta_0\right)$.

For special values, put $A_1=0$, $\mu=0$ and $A_2\neq 0$, the solution comes up

$$u(\eta)=\sqrt{2\sigma}sec\left(\left(-\sqrt{\frac{2(-\mu^2+\lambda^2\sigma)}{\lambda(\mu^2+2\lambda^2\sigma)}}\left(\frac{x^\alpha}{\Gamma(1+\alpha)}+\frac{\gamma t^\alpha}{\Gamma(1+\alpha)}\right)+\eta_0\right)\sqrt{\lambda}\right). \quad (4.1.26)$$

while, we set $A_2=0$, $\mu=0$ and $A_1\neq 0$ then

$$u(\eta)=\sqrt{2\sigma}csc\left(\left(-\sqrt{\frac{2(-\mu^2+\lambda^2\sigma)}{\lambda(\mu^2+2\lambda^2\sigma)}}\left(\frac{x^\alpha}{\Gamma(1+\alpha)}+\frac{\gamma t^\alpha}{\Gamma(1+\alpha)}\right)+\eta_0\right)\sqrt{\lambda}\right). \quad (4.1.27)$$

Type 3: when $\lambda=0$ (Rational function solution)

Using similar process, as mentioned in annotation 1, substituting the values of $u(\eta)$ from eq. (4.1.3), into eq. (4.1.2), along with eq. (3.3) and (3.9), the left hand side of eq. (4.1.2) becomes a polynomial in ϕ and ψ. Taking each coefficient of the polynomial equal to zero, obtain a system of algebraic equations (for the sake of simplicity and shortened, the equations are not represent here) for a_0, a_1, b_1, μ, λ and σ. Solving these system of equations and obtaining the following sets of results

Family 7:

$$a_0=\frac{-\sqrt{3}A_1^2+2\sqrt{3}\mu A_2}{3(-A_1^2+2\mu A_2)},\ a_1=0,\ b_1=-\frac{2(-A_1^2+2\mu A_2)}{\sqrt{3}\mu},\ k=-\frac{\sqrt{\frac{2}{3}}\sqrt{A_1^2-2\mu A_2}}{\mu} \text{ and } c=-\frac{\sqrt{\frac{2}{3}}\gamma\sqrt{A_1^2-2\mu A_2}}{\mu}. \quad (4.1.28)$$

Now, for the above values the new exact wave solution of eq. (1.1) is

$$u(\eta)=\frac{2\mu\eta A_1+4A_1^2+\mu(\mu\eta^2-6A_2)}{\sqrt{3}\mu(\mu\eta^2+2\eta A_1+2A_2)}, \quad (4.1.29)$$

wherein $\eta=\left(-\sqrt{\frac{\frac{2}{3}(A_1^2-2\mu A_2)}{\mu^2}}\left(\frac{x^\alpha}{\Gamma(1+\alpha)}+\frac{\gamma t^\alpha}{\Gamma(1+\alpha)}\right)+\eta_0\right)$.

Specifically, if we take $A_1=0$, $\mu=1$ and $A_2\neq 0$, the bell type solution is

$$u(\eta)=\frac{1}{\sqrt{3}}\left(1-\frac{8}{\left(-\sqrt{\frac{\frac{2}{3}(A_1^2-2\mu A_2)}{\mu^2}}\left(\frac{x^\alpha}{\Gamma(1+\alpha)}+\frac{\gamma t^\alpha}{\Gamma(1+\alpha)}\right)+\eta_0\right)^2+2}\right). \quad (4.1.30)$$

On the other hand, $A_2=0$, $\mu=1$ and $A_1\neq 0$, then the solution is

$$u(\eta)=\frac{1}{\sqrt{3}}\left(1+\frac{4}{\left(-\sqrt{\frac{\frac{2}{3}(A_1^2-2\mu A_2)}{\mu^2}}\left(\frac{x^\alpha}{\Gamma(1+\alpha)}+\frac{\gamma t^\alpha}{\Gamma(1+\alpha)}\right)+\eta_0\right)\left(2+\left(-\sqrt{\frac{\frac{2}{3}(A_1^2-2\mu A_2)}{\mu^2}}\left(\frac{x^\alpha}{\Gamma(1+\alpha)}+\frac{\gamma t^\alpha}{\Gamma(1+\alpha)}\right)+\eta_0\right)\right)}\right). \quad (4.1.31)$$

Family 8:

$$a_0=\frac{\frac{A_1^2}{\sqrt{3}}-\frac{2\mu A_2}{\sqrt{3}}}{-A_1^2+2\mu A_2},\ a_1=0,\ b_1=\frac{2(-A_1^2+2\mu A_2)}{\sqrt{3}\mu},\ k=\frac{\sqrt{\frac{2}{3}}\sqrt{A_1^2-2\mu A_2}}{\mu}\text{ and } c=\frac{\sqrt{\frac{2}{3}}\gamma\sqrt{A_1^2-2\mu A_2}}{\mu}, \quad (4.1.32)$$

Using above results, exact wave solution of eq. (1.1) becomes

$$u(\eta)=-\frac{2\mu\eta A_1+4A_1^2+\mu(\mu\eta^2-6A_2)}{\sqrt{3}\mu(\mu\eta^2+2\eta A_1+2A_2)}. \quad (4.1.33)$$

wherein $\eta=\left(\frac{\sqrt{\frac{2}{3}(A_1^2-2\mu A_2)}}{\mu}\left(\frac{x^\alpha}{\Gamma(1+\alpha)}+\frac{\gamma t^\alpha}{\Gamma(1+\alpha)}\right)+\eta_0\right)$.

Specifically, if we take $A_1=0$, $\mu=-1$ and $A_2\neq 0$, the solution take place as

$$u(\eta)=-\frac{1}{\sqrt{3}}\left(1+\frac{8}{\left(\frac{\sqrt{\frac{2}{3}(A_1^2-2\mu A_2)}}{\mu}\left(\frac{x^\alpha}{\Gamma(1+\alpha)}+\frac{\gamma t^\alpha}{\Gamma(1+\alpha)}\right)+\eta_0\right)^2-2}\right). \quad (4.1.34)$$

On the other hand, $A_2=0$, $\mu=1$ and $A_1\neq 0$, then the kink type solution is

$$u(\eta)=-\frac{1}{\sqrt{3}}\left(1+\frac{4}{\left(\frac{\sqrt{\frac{2}{3}(A_1^2-2\mu A_2)}}{\mu}\left(\frac{x^\alpha}{\Gamma(1+\alpha)}+\frac{\gamma t^\alpha}{\Gamma(1+\alpha)}\right)+\eta_0\right)\left(\left(\frac{\sqrt{\frac{2}{3}(A_1^2-2\mu A_2)}}{\mu}\left(\frac{x^\alpha}{\Gamma(1+\alpha)}+\frac{\gamma t^\alpha}{\Gamma(1+\alpha)}\right)+\eta_0\right)+2\right)}\right). \quad (4.1.35)$$

Remark 1: All of the mentioned above solutions have been checked by substituting back into the original equation via the symbolic computer software package program Mathematica and emerged them purely.

5. Graphical representation of our solutions

In this section, we explained physical illustration and represent various shapes of three dimensional corresponding with two dimensional graphical patterns for the acquired solutions of space-time fractional equation. Also, we describe the shape of contour plot of our attained solutions. These solutions are of hyperbolic, trigonometric and rational function solutions. Moreover, by taking several free parameters, the exact solutions of these equations are converted into various wave shapes of kink, singular kink, bell, periodic, soliton and multiple soliton respectively. Hereafter, the solution of eq. (4.1.6) propound in Fig. (1), it depicts the kink shape solution with $\lambda = 1.1$, $\sigma = 0.6$, $\gamma = 0.5$, $\eta_0 = 0.8$ within the interval $-2.5 \le x \le 5.5$, $-3.5 \le t \le 4.5$. The solution of eq. (4.1.7) imprint the multiple soliton solution for the values of λ=0.5, $\sigma = 1.2$, $\gamma = 0.2$, $\eta_0 = 0.2$, $-1.5 \le x \le 2.5$ and $-3.5 \le t \le 4.5$, it shows is in Fig. (2). Taking different values of free parameters the solution of eq. (4.1.10), (4.1.14), (4.1.30) represented is in Fig. (3), (5), (10), which denominate the bell shape solution within the interval $-2.5 \le x \le 3.5$, $-2.5 \le t \le 1.5$ and $-3.5 \le x \le 4.5$, $-1.5 \le t \le 3.5$ and $-3.5 \le x \le 4.5$, $-2.5 \le t \le 3.5$ respectively. Also, the solution of eq. (4.1.11) and (4.1.15) delineate in Fig. (4) and (6), which gives the singular soliton shape with $\lambda = 0.5$, $\sigma = 9.4$, $\gamma = -0.2$, $\eta_0 = -0.4$ and $\lambda = 0.2$, $\sigma = 5.2$, $\gamma = -0.8$, $\eta_0 = -2.3$, $\mu = 0.2$ within intervals $-3.5 \le x \le 4.5$, $-2.5 \le t \le 3.5$ and $-2.5 \le x \le 4.5$, $-3.5 \le t \le 4.5$ respectively. On the other hand, Fig. (7), (8), (9) describe the periodic type solution with special values of $\lambda = 1.4$, $\sigma = 0.3$,$\gamma = 2.5$, $\eta_0 = 0.2$ and $\lambda = 0.4$, $\sigma = 2.3$, $\gamma = -0.5$, $\eta_0 = -0.8$ and $\lambda = 8.5$, $\sigma = 1.2$, $\gamma = 0.5$, $\eta_0 = 0.4$, $\mu = 0.5$ within intervals $-1.5 \le x \le 3.5$, $-2.5 \le t \le 4.5$ and $-3.5 \le x \le 3.5$, $-3.5 \le t \le 4.5$ and $-2.4 \le x \le 4.4$, $-1.5 \le t \le 3.5$ respectively. At last, the solution of eq. (4.1.35) delineate in Fig. (11), which gives the singular kink type wave solution with special values $\gamma = 0.1$, $\eta_0 = 0.2$ within intervals $-2.5 \le x \le 4.5$ and $-1.5 \le t \le 3.5$.

(a) (b) (c)

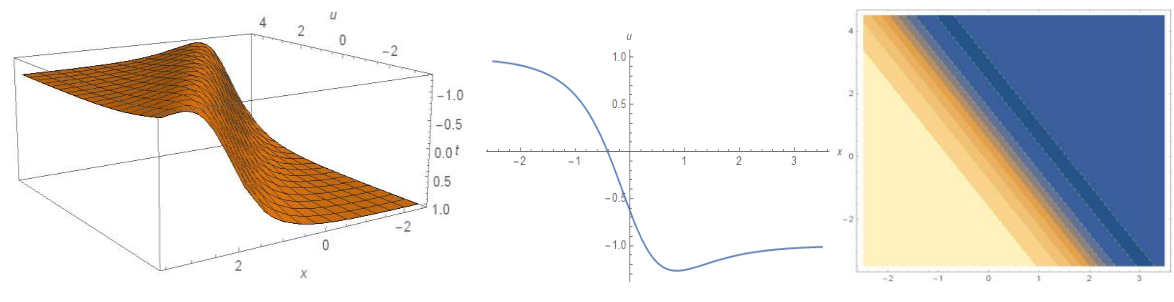

Figure1: Plot of eq. (4.1.6) corresponding to the values (a) $\lambda = 1.1$, $\sigma = 0.6$, $\gamma = 0.5$, $\eta_0 = 0.8$ with its projection at (b) $t = 1$and (c) represents contour plot.

(a) (b) (c)

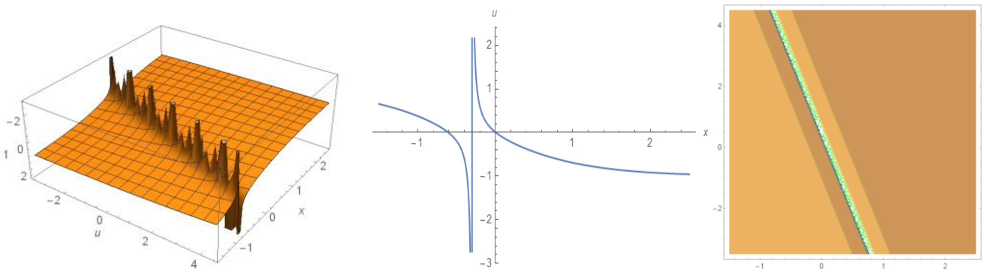

Figure 2: Plot of eq. (4.1.7) corresponding to the values (a) λ=0.5, $\sigma = 1.2$, $\gamma = 0.2$, $\eta_0 = 0.2$ with its projection at (b) $t = 2$and (c) represents contour plot.

(a) (b) (c)

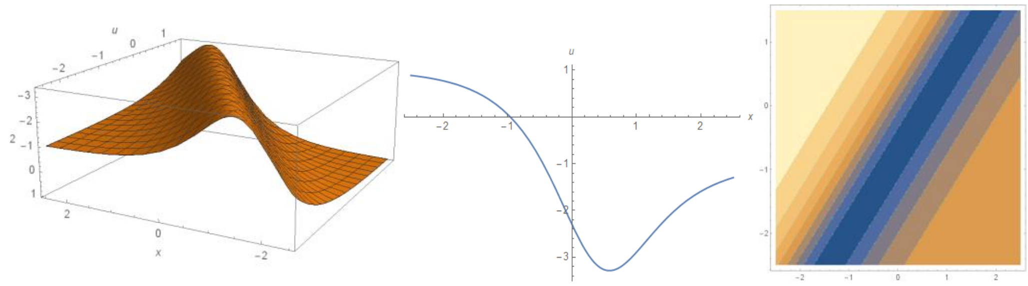

Figure 3: Plot of eq. (4.1.10) corresponding to the values (a) $\lambda = 0.4$, $\sigma = 9.8$, $\gamma = -0.8$, $\eta_0 = -0.8$ with its projection at (b) $t = 0$ and (c) represents contour plot.

(a) (b) (c)

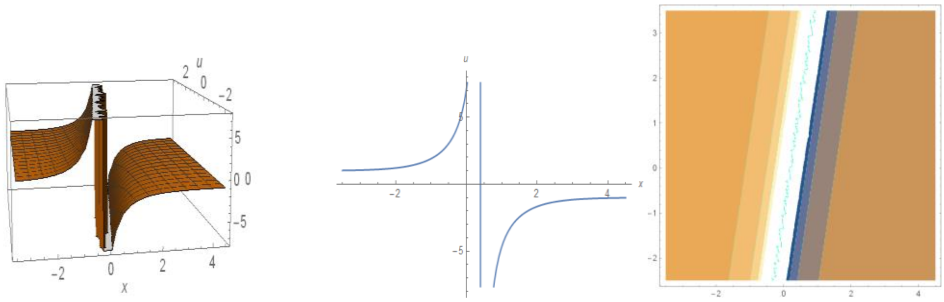

Figure 4: Plot of eq. (4.1.11) corresponding to the values (a) $\lambda = 0.5$, $\sigma = 9.4$, $\gamma = -0.2$, $\eta_0 = -0.4$ with its projection at (b) $t = 1$and (c) represents contour plot.

(a) (b) (c)

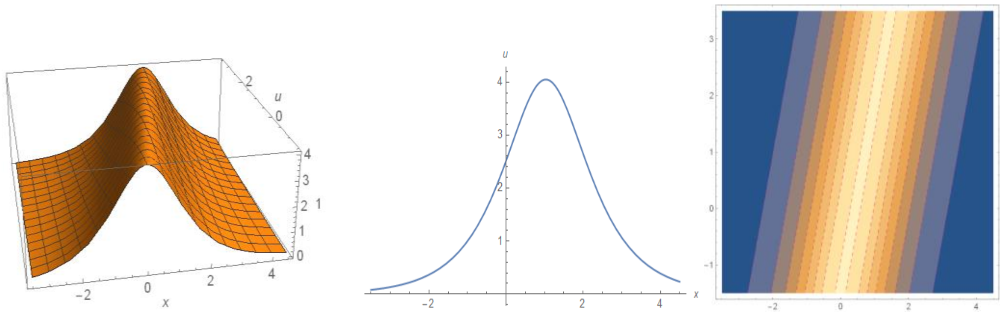

Figure 5: Plot of eq. (4.1.14) corresponding to the values (a) $\lambda = 0.4$, $\sigma = 8.2$,$\gamma = -0.3$, $\eta_0 = -0.7$, $\mu = 0.2$ with its projection at (b) $\mathrm{t} = 2$ and (c) represents contour plot.

(a) (b) (c)

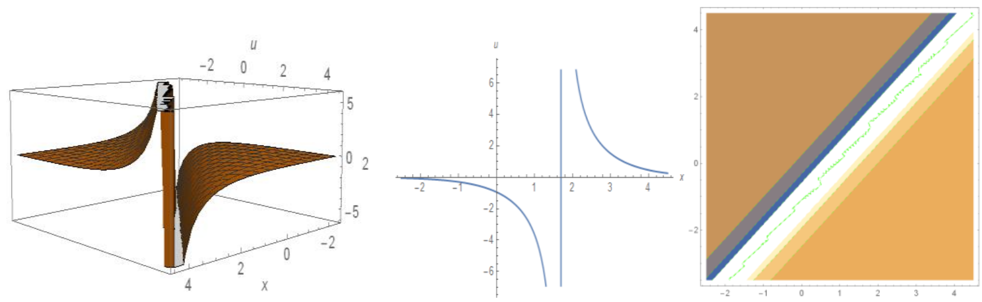

Figure 6: Plot of eq. (4.1.15) corresponding to the values (a) $\lambda = 0.2$, $\sigma = 5.2$, $\gamma = -0.8$, $\eta_0 = -2.3$, $\mu = 0.2$ with its projection at (b) $\mathrm{t} = 1$ and (c) represents contour plot.

(a) (b) (c)

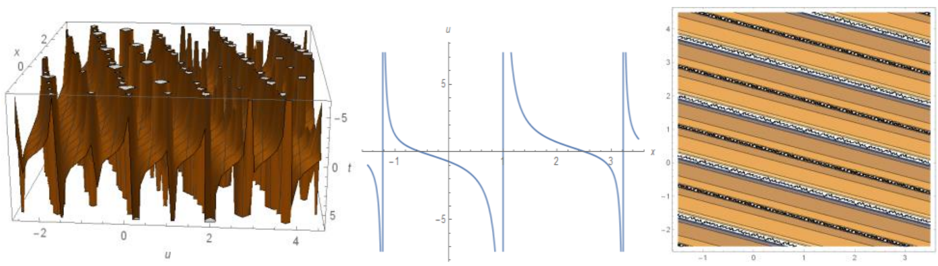

Figure 7: Plot of eq. (4.1.18) corresponding to the values (a) $\lambda = 1.4$, $\sigma = 0.3$,$\gamma = 2.5$, $\eta_0 = 0.2$ with its projection at (b) $\mathrm{t} = 1$ and (c) represents contour plot.

(a) (b) (c)

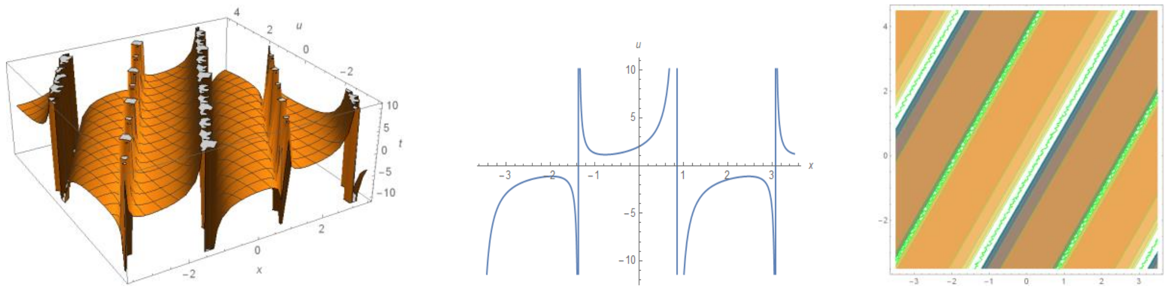

Figure 8: Plot of eq. (4.1.23) corresponding to the values (a) $\lambda = 0.4$, $\sigma = 2.3$, $\gamma = -0.5$, $\eta_0 = -0.8$ with its projection at (b) $\mathrm{t} = 1$and (c) represents contour plot.

(a) (b) (c)

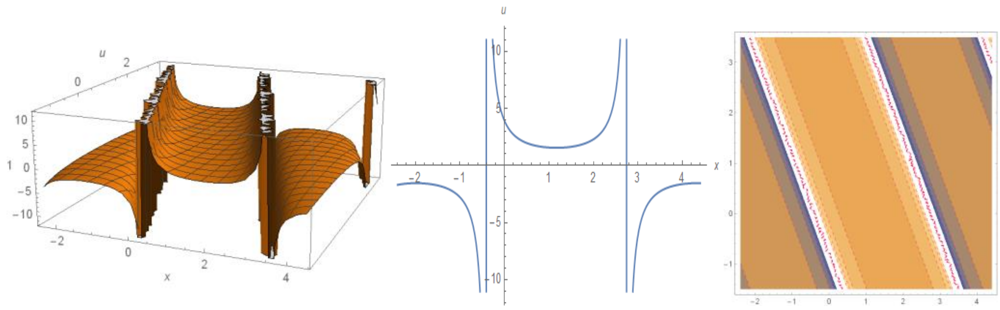

Figure 9: Plot of eq. (4.1.26) corresponding to the values (a) $\lambda = 8.5$, $\sigma = 1.2$, $\gamma = 0.5$, $\eta_0 = 0.4$, $\mu = 0.5$ with its projection at (b) $\mathrm{t} = 0$ and (c) represents contour plot.

(a) (b) (c)

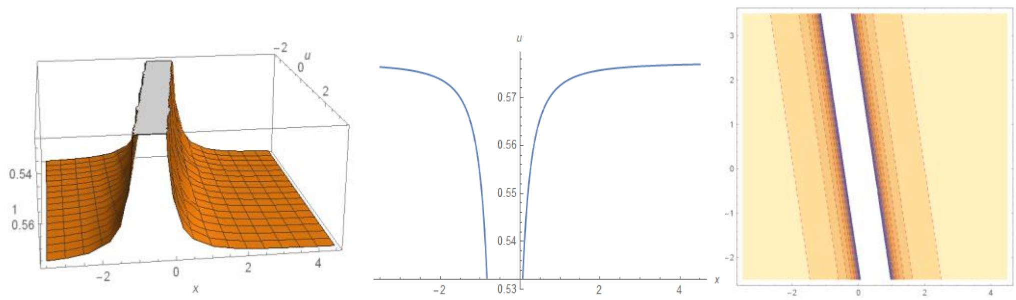

Figure 10: Plot of eq. (4.1.30) corresponding to the values (a) $\gamma = 0.5$, $\eta_0 = 0.8$, $\mu = 0.2$ with its projection at (b) $\mathrm{t} = 2$ and (c) represents contour plot.

(a) (b) (c)

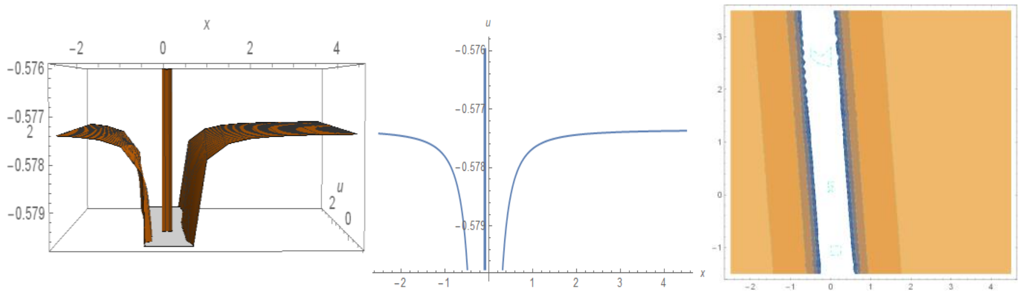

Figure 11: Plot of eq. (4.1.35) corresponding to the values (a) $\gamma = 0.1, \eta_0 = 0.2$ with its projection at (b) $t = 1$ and (c) represents contour plot.

6. Results and Discussion

The principal motive of an advanced method called the two variable $(G'/G, 1/G)$-expansion method is to accentuate new and further general and various existing wave solutions of the above-mentioned equations (1.1). In our gained solutions, as the two parameters A_1 and A_2 receive various special values, various types of traveling wave solution converts into the solitary wave solution. Whereas $\mu = 0$ and $b_j = 0$ in eqns (3.1) and (3.13), the two variable $(G'/G, 1/G)$-expansion method transform into the original (G'/G)-expansion method. In Ref. [27], the solution of space-time fractional Cahn Hillihard equation has been investigated and attains some sets of solution, which is in the forms of hyperbolic (tan, cot) function solutions and plotted only 3D shape of their solutions. But utilizing $(G'/G, 1/G)$-expansion method in this paper, we obtain twenty four solutions and these are represented is in the forms of hyperbolic (tanh,coth,sech and csch), trigonometric (tan, cot,sec,csc) and rational function solutions. Also, setting different particular values of the parameters multiple soliton, kink, singular kink, periodic and compacton solutions of Cahn Hillihard equation are found. Moreover, these shapes depicts is in the form of 2D, 3D along with contour plot of our attained solutions, which also have not been reported in the previous literature.

7. Conclusion

In this paper, recently developed two variables $(G'/G, 1/G)$-expansion methods has been successfully imputed for analyzing new and further more exact traveling and solitary wave solutions for a class of nonlinear space time fractional Cahn Hillihard equation. Hereafter, by taking various arbitrary values of free parameters, the solutions converts into different shapes of wave solutions as like as soliton, periodic etc. which are presented graphically. To the best of our knowledge, we are assured that the solutions remain its uniformity upon interacting with others. The worked out results stated that the current method provides a powerful mathematical instrument that concise computational complication, efficient and can be executive approach from the theoretical and practical point of view. In all respects, it is unavoidable to notice that this method can be more frequently applicable to solve different nonlinear fractional differential equations along with NLEEs which regularly emerges in the arena of nonlinear sciences.

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