International Journal of Sciences: Basic and Applied Research (IJSBAR)

ISSN 2307-4531
(Print & Online)

http://gssrr.org/index.php?journal=JournalOfBasicAndApplied

Two Conditional proofs of Riemann Hypothesis

Jamal Y. Mohammad Salah*

Department of Basic Science, College of Health and Applied Sciences, A'Sharqiyah University, Ibra post code 400 Oman
Email: damous73@yahoo.com

Abstract

We consider the analytic continuity of the Riemann zeta function through the Hankel contour. We detect a sort of non accuracy in the functional equation with a significantly small error that we consider to conditionally prove Riemann Hypothesis in two ways.

Keywords: Riemann Hypothesis; Analytic Continuity; Functional Equation; Hankel Contour.

1. Introduction

The Riemann zeta function is a specific form of Dirichlet series defined initially for complex *s* with real part greater than 1 by the absolutely convergent infinite series

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$$

Early in the 1730s Leonhard Euler successfully proved a product representation of $\zeta(s)$ for real values

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}} = \frac{1}{1 - p^{-2}} \cdot \frac{1}{1 - p^{-3}} \cdot \frac{1}{1 - p^{-5}} \cdots$$

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* Corresponding author.

Where the infinite product extends over all prime numbers p [1]. The Riemann hypothesis explores zeros outside the region of convergence of this series and Euler product. The analytic continuation of the function is necessary to obtain a form that is valid for all complex s. Since the zeta function is meromorphic, the one can provide an analytic continuation to be unique and functional forms equivalent over their domains. In fact the zeta function can be extended to give a finite value for all values of s except for the simple pole at $s = 1$. Consequently, the zeta function satisfies the functional equation

$$\zeta(s) = 2^s \pi^{s-1} \, sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\, \zeta(1-s)$$

Therefore, if s is a negative even integer then $\zeta(s)= 0$ because the factor $sin\left(\frac{\pi s}{2}\right)$ vanishes; these are the *trivial zeros* of the zeta function. The functional equation also implies that the zeta function has no zeros with negative real part other than the trivial zeros, so all non-trivial zeros lie in the critical strip where s has real part between 0 and 1.

In 1859, Riemann introduced s the function of the complex variable t defined by

$$\xi(s) = \frac{1}{2}\, s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s),$$

With $s = \frac{1}{2} + it$, showing that $\xi(t)$ is an even entire function of t whose zeros have imaginary part between $-\frac{i}{2}$and $-\frac{i}{2}$. He further states that in the range between 0 and T the function $\xi(t)$has about $\left(\frac{T}{2\pi}\right)\ln\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi}$ zeros. Riemann then continues: "Indeed, one finds between those limits about that many real zeros, and it is very likely that all zeros are real." The statement that all zeros of the function $\xi(t)$ are real is the Riemann hypothesis. Beside the trivial zeros the function $\zeta(s)$ has non-trivial zeros that are the complex numbers $\frac{1}{2} + \alpha\, i$ where α a zero of $\xi(t)$. Thus, in terms of the function $\zeta(s)$, Riemann hypothesis (RH) is: The nontrivial zeros of $\zeta(s)$, have real part equal to $\frac{1}{2}$

General observations

The motivation of our work is relevant to the following observations: for any complex number s

$$t^{-s} + t^{s-1} = 0 \Longrightarrow t = \frac{1}{2} + i\,\frac{(2m+1)\pi}{2lnt}, \qquad m \in \mathbb{Z}, \qquad t \neq 0 \text{ and } lnt \neq 0.$$

$$\zeta(s) + \zeta(1-s) = \sum_{n=1}^{\infty} n^{-s} + n^{s-1}$$

If the right hand side equals zero then it is possible that $\zeta(s) = \zeta(1-s) = 0$.

Although $\sum_{n=1}^{\infty} n^{-s} + n^{s-1} = 0$ requires that $n^{-s} + n^{s-1} = 0$ for every n, that is $y = \frac{(2m+1)\pi}{2lnn}$ for every n which is *impossible* but it gives a hint regarding the location of zeroes on the critical line $Re(s) = \frac{1}{2}$.

.An elementary approach

For a sufficiently large N

$$\zeta(s) - \sum_{n \neq N}^{\infty} n^{-s} = N^{-s}$$

$$\zeta(1-s) - \sum_{n \neq N}^{\infty} n^{s-1} = N^{s-1}$$

If we add the last two expressions

$$\zeta(s) - \sum_{n \neq N}^{\infty} n^{-s} + \zeta(1-s) - \sum_{n \neq N}^{\infty} n^{s-1} = N^{-s} + N^{s-1}.$$

Now, if $\zeta(s) = \zeta(1-s) = 0$, then most probably (*not accurate*) and at infinity

$$\sum_{n \neq N}^{\infty} n^{-s} = \sum_{n \neq N}^{\infty} n^{s-1} = 0.$$

If that holds true, then the right hand side also equals zero, that is $Re(s) = \frac{1}{2}$.

From the observations above; we presume that

$$\zeta(s) = \zeta(1-s) = 0,$$

is an *assumption*, and the *location* is determined by the solution of $t^{-s} + t^{s-1} = 0,\ t \neq 0$.

Hypothetically, we can view zeta zeroes via two concepts: *location* and *quantity*, the location is imposed by some structures as per zeta function definition, and then the quantity is to be computed.

The main claim of our work is that there exists a sort of *approximation* in all aspects of zeta function except the *location* of its non-trivial zeros subject to their existence. For that we state and prove the first two results that are directly relevant to our aim. For the sake of simplicity we provide a brief proof of the analytic continuity and the first functional equation since the results are widely- known. Then we assume some justified modifications so that the main conclusion “Conditional Proof of Riemann Hypothesis” is achieved logically.

Riemann Zeta FunctionIntegral Formula and First Functional Equation

Lemma 1*the Riemann Zeta function is meromorphic everywhere, except at a simple pole s = 1*

Proof

$$\Gamma(s) = \int_0^\infty e^{-\tau} \tau^{s-1} d\tau = n^s \int_0^\infty e^{-nt} t^{s-1} dt, \quad \tau = nt.$$

Multiplying by $\zeta(s)$, implies

$$\zeta(s)\Gamma(s) = \int_0^\infty \frac{t^{s-1}}{e^t - 1}\, dt \;, \quad Re(s) > 0.$$

To extend this formula to $\mathbb{C}\backslash\{1\}$, we integrate$(-t)^s/(e^t-1)$ over a Hankel contour: a path from $+\infty$ inbound along the real line to $\epsilon > 0$, counterclockwise around a circle of radius ϵ at 0, back to ϵ on the real line, and outbound back to $+\infty$ along the real line, around the circle, t can be parameterized by $t = \epsilon e^{i\theta}, 0 \le \theta \le 2\pi$ and ϵ is a small arbitrary positive constant that we will let tend to 0:

$$\oint_C \frac{(-t)^{s-1}}{e^t - 1}\, dt = \int_{\rho_1} \frac{(-t)^{s-1}}{e^t - 1} dt + \int_{\rho} \frac{(-t)^{s-1}}{e^t - 1} dt + \int_{\rho_2} \frac{(-t)^{s-1}}{e^t - 1} dt$$

$$= \int_R^\epsilon \frac{(te^{-\pi i})^{s-1}}{e^1 - 1}\, dt + i\epsilon \int_0^{2\pi} \frac{\left(\epsilon e^{-\pi i} e^{i\theta}\right)^{s-1}}{e^{\epsilon e^{i\theta}} - 1} e^{i\theta} d\theta + \; e^{2\pi i} \int_\epsilon^R \frac{(te^{-\pi i} e^{2\pi i})^{s-1}}{e^{te^{2\pi i}} - 1} dt$$

$$= \; -e^{-\pi i(s-1)} \int_\epsilon^R \frac{t^{s-1}}{e^t - 1}\, dt + i\epsilon \int_0^{2\pi} \frac{\left(\epsilon e^{-\pi i} e^{i\theta}\right)^{s-1}}{e^{\epsilon e^{i\theta}} - 1} e^{i\theta} d\theta + \; e^{(2\pi i - \pi i)(s-1)} \int_\epsilon^R \frac{t^{s-1}}{e^t - 1} dt$$

$$= \left(e^{\pi i(s-1)} - e^{-\pi i(s-1)}\right) \int_\epsilon^R \frac{t^{s-1}}{e^t - 1}\, dt + i\epsilon \int_0^{2\pi} \frac{\left(\epsilon e^{-\pi i} e^{i\theta}\right)^{s-1}}{e^{\epsilon e^{i\theta}} - 1} e^{i\theta} d\theta$$

$$= -2i\, sin(\pi s) \int_\epsilon^R \frac{t^{s-1}}{e^t - 1}\, dt + i\epsilon \int_0^{2\pi} \frac{\left(\epsilon e^{-\pi i} e^{i\theta}\right)^{s-1}}{e^{\epsilon e^{i\theta}} - 1} e^{i\theta} d\theta$$

$$\lim_{\substack{\epsilon \to 0 \\ R \to \infty}} \left[-2i\, sin(\pi s) \int_\epsilon^R \frac{t^{s-1}}{e^t - 1}\, dt + i\epsilon \int_0^{2\pi} \frac{\left(\epsilon e^{-\pi i} e^{i\theta}\right)^{s-1}}{e^{\epsilon e^{i\theta}} - 1} e^{i\theta} d\theta \right], \quad Re(s) > 1$$

$$= 2i\, sin(\pi s) \int_0^\infty \frac{t^{s-1}}{e^t - 1}\, dt.$$

Finally;

$$\oint_C \frac{(-t)^{s-1}}{e^t - 1}\, dt = 2i\, sin(\pi s)\Gamma(s)\zeta(s) \tag{1.1}$$

Lemma 2

$$\zeta(s) = 2^s \pi^{s-1}\, sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\,\zeta(1-s)$$

Proof

Here we consider a modified Hankel contour: consisting of two circles centered at the origin and a radius segment along the positive reals. The outer circle has radius $(2n+1)\pi$ and the inner circle has radius $\epsilon < \pi$. The outer circle is traversed clockwiseand the inner one counterclockwise. The radial segment is traversed in both directions. Then by employing the residue theorem [2, 7].

$$\oint_\gamma \frac{(-t)^{s-1}}{e^t - 1}\, dt = 2^s \pi^{s-1} sin\left(\frac{\pi s}{2}\right)\sum_{n=1}^{\infty} n^{s-1}$$

Plugging in equation (1.1) we then prove the functional equation

$$\zeta(s) = 2^s \pi^{s-1}\, sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\,\zeta(1-s)$$

Remark 1

From Lemma 1, the integral along the real axis in both directions does not depend on ϵ. Similarly; the integral along the modified Hankel contour in Lemma 2 does not depend on the path. The only significant note is the integral around the small circle vanishes subject to $\epsilon = 0$.

Claim

$\epsilon \nrightarrow 0$.In other words ϵ will remain non-zero no matter how small it is.

Proof

From the substitution $\tau = nt$, since we considered the sum of the geometric series, we already assumed that $n \to \infty$, that is $t \neq 0$ otherwise $\tau = \infty.0$ is an undefined term. Now, since t can be parametrized around the small circle by $t = \epsilon e^{i\theta}$, this implies $|t| = \epsilon \neq 0$

According to our claim, we will keep track on ϵ along the steps of the proofs of Lemma 1 and Lemma 2. We will let the integral around the circle.

$$t = \epsilon e^{i\theta} \Rightarrow i\epsilon \int_{\theta > 0}^{2\pi} \frac{\left(\epsilon e^{-\pi i} e^{i\theta}\right)^{s-1}}{e^{\epsilon e^{i\theta}} - 1} e^{i\theta} d\theta$$

$$i\epsilon \int_{\theta > 0}^{2\pi} \frac{\left(\epsilon e^{-\pi i} e^{i\theta}\right)^{s-1}}{e^{\epsilon e^{i\theta}} - 1} e^{i\theta} d\theta \rightarrow f(s,\epsilon)\epsilon^{s-1} \rightarrow \epsilon^{s-1}$$

For shorthand and since the function $f(s,\epsilon)$ will not directly contribute in our approach, we can simply omit it. Now, we consider the slight changes on Lemma 1 and Lemma 2, the result in (1.1) will reduce to

$$\oint_C \frac{(-t)^{s-1}}{e^t - 1}\, dt = \epsilon^{s-1} + 2i\, sin(\pi s)\Gamma(s)\zeta(s),$$

Similarly, the functional equation in Lemma 2 will be also modified and viewed as *non functional equations*. Since we assumed the additional terms of ϵ we have to add restrictions for accuracy.

$$\zeta(s) = \epsilon^{s-1} + 2^s \pi^{s-1} sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\, \zeta(1-s), \quad Re(s) > 1, \tag{1.2}$$

$$\zeta(1-s) = \epsilon^{-s} + 2^{1-s}\pi^{-s} sin\left(\frac{\pi(1-s)}{2}\right)\Gamma(s)\,\zeta(s), \quad Re(s) < 0 \tag{1.3}$$

In the critical strip we multiply (1.2) and (1.3) by ϵ

$$\epsilon\zeta(s) = \epsilon^s + \epsilon.2^s\pi^{s-1} sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\,\zeta(1-s), \quad 0 < Re(s) < 1 \tag{1.4}$$

$$\epsilon\zeta(1-s) = \epsilon^{1-s} + \epsilon.2^{1-s}\pi^{-s} sin\left(\frac{\pi(1-s)}{2}\right)\Gamma(s)\,\zeta(s), \quad 0 < Re(s) < 1 \tag{1.5}$$

Theorem 1" *The first Conditional Proof of Riemann Hypothesis"*

If $\zeta(s) = \zeta(1-s) = 0$, *then* $Re(s) = \frac{1}{2}$

Proof

In general and without assuming the location of the zeros, that is we omit the restriction in (1.2) and (1.3) and we ignore *the non accuracy*. For simplicity we write

$$A(s) = 2^s\pi^{s-1} sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\,\zeta(1-s)$$

$$A(1-s) = 2^{1-s}\pi^{-s} sin\left(\frac{\pi(1-s)}{2}\right)\Gamma(s)\,\zeta(s)$$

Let $s = x + iy$ satisfying $\zeta(s) = \zeta(1-s) = 0$,then

$$\zeta(s) - A(s)\zeta(1-s) = \epsilon^{s-1}$$

$$\zeta(1-s) - A(1-s)\zeta(s) = \epsilon^{-s}.$$

Adding the last two expressions yield

$$\zeta(s) - A(s)\zeta(1-s) + \zeta(1-s) - A(1-s)\zeta(s) = \epsilon^{s-1} + \epsilon^{-s}$$

Now, if the left hand side of the expression above equals zero implies the right hand side also equals zero, solving for $s \in \mathbb{C}\backslash\{1\}$:

$$\epsilon^{s-1} + \epsilon^{-s} = 0$$

$$y = \frac{i\, ln\epsilon\, (2x-1) + (2m+1)\pi}{2ln\epsilon}, \quad \epsilon \neq 0, \quad ln\epsilon \neq 0 \text{ and } m \in \mathbb{Z}$$

Since y is real valued then the term multiplied by i will vanish that is $x = \frac{1}{2}$, or if we write

$$s = x + i\left[\frac{i\, ln\epsilon\, (2x-1) + (2m+1)\pi}{2ln\epsilon}\right]$$

The real part of s will reduce to $x = \frac{1}{2}$.

In particular; if we consider the critical strip we consider (1.4) and (1.5). We deduce

$$\epsilon^{s} + \epsilon^{1-s} = 0 \rightarrow Re(s) = \frac{1}{2}$$

Note that $\epsilon \neq 0$ plays a crucial role in the result. Now, since we assumed that $\zeta(1-s) = 0$ then zeta zeroes are of the form:

$$s = \frac{1}{2} \pm i\left(\frac{(2m+1)\pi}{2ln\epsilon}\right), \epsilon \neq 0, ln\epsilon \neq 0 \; and \; m \in \mathbb{Z}$$

Next; we consider another approximation from Lemma 1:

Claim

$$\zeta(s)\Gamma(s) \neq \int_{0}^{\infty} \frac{t^{s-1}}{(e^{t}-1)}\, dt$$

This is due to the same reason; $n \rightarrow \infty$ (already assumed by considering the sum of the geometric series, that is $t \neq 0$, in details:

By assumption $\tau = nt$, since τ varies from 0 to ∞ that is:

When $\tau = 0$ then: $t = 0$ and $n < \infty$(finite).

When $\tau \to \infty$ then: either $(t \text{ and } n \to \infty)$ or $(t \to \infty, n < \infty)$, or $(n \to \infty \text{ and } t \neq 0)$.

The geometric series is then by using *t* once only (not twice as assumed in Lemma 1)

$$\sum_{n=1}^{\infty} e^{-nt} = \sum_{\substack{t=0 \\ n<\infty}} e^{-nt} + \sum_{\substack{t>0 \\ n=1}}^{\infty} e^{-nt}$$

$$\sum_{n=1}^{\infty} e^{-nt} t^{s-1} = \sum_{\substack{t=0 \\ n<\infty}} 0 + \sum_{\substack{t>0 \\ n=1}}^{\infty} e^{-nt} t^{s-1}$$

Thus

$$\zeta(s)\Gamma(s) = \int_{t>0}^{\infty} \frac{t^{s-1}}{(e^t - 1)}\, dt, \qquad n \; already \; \to \infty$$

Now;

$$\zeta(s)\Gamma(s) = \int_{t>0}^{\infty} \frac{t^{s-1}}{(e^t - 1)}\, dt, \quad Re(s) > 1 \tag{1.6}$$

$$\zeta(1-s)\Gamma(1-s) = \int_{t>0}^{\infty} \frac{t^{-s}}{(e^t - 1)}\, dt, \quad Re(s) < 0 \tag{1.7}$$

Theorem 2 *"The second conditional proof of Riemann Hypothesis"*

If $\zeta(s) = \zeta(1-s) = 0$ *Then* $Re(s) = \frac{1}{2}$

Proof

We rearrange (1.6) and (1.7) and omit the restrictions by assuming *the analytic continuity*, we have

$$\zeta(s)\Gamma(s) + \zeta(1-s)\Gamma(1-s) = \int_{t>0}^{\infty} \frac{t^{s-1} + t^{-s}}{(e^t - 1)}\, dt,$$

The integral at the right will then vanish element wise

$$t^{s-1} + t^{s} = 0, \quad t \neq 0.$$

This implies $Re(s) = \frac{1}{2}$.

2. Conclusion

Gamma function inaccurately completes Zeta function. This is due to the undefined term $\infty.0$ that will cause a gap no matter how small it is. This gap will then result to a sufficiently small error (*invisible*) that we can detect by considering $\zeta(s) + \zeta(1-s)$ as a block. The non-trivial Zeros as *assumed,* if they exist they are certainly located on the critical line due to the coincidence to the zeroes of structures of the form $t^{s-1} + t^{-s} = 0$. That is the nontrivial Zeroes are connected to the error zeroes. We may then consider Riemann Hypothesis as a true assumption subject to approximations, for that we deduce:

$$\zeta(s) = 2^{s}\pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\, \zeta(1-s) \text{ is accurate but not } sharp,$$

$$\zeta(-2n) = 0, \text{is accurate but not } sharp,$$

$$\zeta\left(\frac{1}{2} + it\right) = 0, \text{for some t is accurate but not } sharp.$$

The only sharp result is the location of non trivial zeroes if assumed to exist.

3. Funding Statement

The author received no financial support for the research, and publication of this article

Acknowledgments

I express my gratitude to the Almighty for giving me strength and courage during the process of research writing and completion of this piece of work. I extend my gratitude to all researchers who provided a holistic approach to understand Riemann Hypothesis. The acknowledgments cannot be complete without thanking Stephen Wolfram and all Wolfram Alpha team for their outstanding website: www.wolframalpha.com

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