International Journal of Sciences: Basic and Applied Research (IJSBAR)

ISSN 2307-4531
(Print & Online)

http://gssrr.org/index.php?journal=JournalOfBasicAndApplied

The Structure Theorems for Infinite Abelian Groups

Boyd Mwalungali[a]*, William.Sakala[b]

[a, b] *Department of Mathematics Copperbelt University 21692 Kitwe, Zambia*

[a]*Email: mwalungalib@yahoo.com*

[b]*Email: sakalawilliam@yahoo.com*

Abstract

This paper, examines one of the most fundamental and interesting algebraic structures, infinite abelian groups, from the perspective of group theory The Theory of abelian groups is generally simpler than that of their non-abelian counterparts and finite abelian groups are very well understood. Then would like to state whether the theory of Structure for infinite abelian group of additive rational number is finitely generated or a divisible abelian group and also show examples of each classification. Thus, classify groups by stating whether elements are finite or infinite and categories them as infinite abelian Groups. Moreover, an application to homology group and rotation in two dimensions is presented, as a demonstration of the structure of infinite abelian groups utility.

Keywords: groups; finite and infinite abelian groups; finitely generated groups; divisible groups.

1. Introduction

The source of finite abelian group theory is grounded mainly in number theory and in the theory of quadratic forms. The emergence of a theory of finite abelian groups was first discernible in the late eighteenth and early nineteenth centuries in the arguments of Euler, Lagrange, and Gauss. Despite the fact that groups were not yet defined and that it needed over half a century before these creatures got a name. It was Kronecker in 1870 who introduced in the notion of an abstract abelian group. Working with radical field extensions it must have been natural to him to name commutative groups after the Norwegian mathematician Nils Henrik Abel whose 200th birthday commemoration was celebrated at Oslo University during that year. Kronecker also re-proved **Gauß's theorem** for finite abelian groups and provided a primary decomposition for these groups, see van der Waerden [13: 149].

---

* Corresponding author.

Van der Waerden [13: 149] also mentions that another proof of Gauß's theorem appears in Abel's work and Schering, a pupil of Gauß, published a paper about binary quadratic forms in which he proves Gauß's theorem for finite abelian groups. When the theory of groups was first introduced, the attention was on finite groups. Now, the infinite abelian groups have come into their own. The results obtained in infinite abelian groups are very interesting and penetrating in other branches of Mathematics. This paper presents the most significant solutions in infinite abelian groups following the discovering given by Frank Ayres in his Schaum's Outlines of abstract algebra and J Rotman in his book, Theory of Groups. In order to facilitate our study, infinite abelian groups are used. The study of abelian groups concerning the study of torsion groups and torsion-free groups and also fundamental theorems of finitely generated abelian groups has been defined and shown. Then also the study of divisible groups. It is assumed that the reader is familiar with elementary group theory and finite abelian groups which will look into briefly with minor proofs accompanied with some motivational statement. However, the fundamental theorem of abelian groups has only been stated with examples and not proved. Finally, the result of structure theorem of infinity abelian groups can be shown by application in infinite dimensional of lie groups and computational of homology groups in algebraic topology.

1.1 Definitions and Notation

This section introduces some basic definitions in group theory. This includes groups, subgroups and normality. Then use multiplicative notation to represent the binary operation on the group .In addition, also show some examples to illustrate these terms. First and foremost, state the meanings of several shorthand symbols.

• **|:** divides.

• **∀:** for all.

• **∃:** exists.

• **∈:** element of.

• **⇒:** implies.

• **R:** the set of real numbers.

• **A×B:** Cartesian product of sets A and B.

• **Z:** the set of the integers.

•$\boldsymbol{Z_n}$**:** the set of the integers modulo n. Example: {0, 1, 2} for $\boldsymbol{Z_3}$

• ≅ Isomorphic. Example: The Klein-4 group ≅ $Z_2 \times Z_2$.

•$\langle \boldsymbol{x} \rangle$**:** the orbit of x; that is, the set of all powers of x.

• **|S|:** The size of set S. Furthermore, this differentiates between different types of ∞, but the largest sets discussed here are countably infinite. Example :$|\{0, 1, 2\}| = 3$.

A **function** is a mapping F from set A to set B, where each input from A has exactly one output in B. The following are some important properties used throughout this thesis.

• **1-1:** a function f: X → Y is one to one if ∀ x, y ∈ X, f(x) = f(y) ⇒ x = y.

• **Onto:** a function f: X → Y is onto if ∀ y ∈ Y, ∃ x ∈ X such that f(x) = y.

• **Bijection**: a function f: x → y is a bijection if it is both 1-1 and onto.

The following table is useful in "translating" multiplication notation into additive notation, a and b are elements of a group G.H and K are subgroups of G.

Table 1

Multiplicative	ab	a^{-1}	1	a^n	ab^{-1}	HK	aH
Additive	$a+b$	$-a$	0	na	$a-b$	$H+K$	$a+H$

1.2 Groups

The main goal is to review some fundamentals in group theory. This will include basic definitions, structures of subgroups and direct products, and isomorphism, all of which will be integral in analyzing both finite and infinite abelian groups.

Definition 1.1 Binary Operations

A binary operation on a set S is a function that maps two inputs from S back into S. Expressed $\langle s,*\rangle$, a more formal definition is a function $*: S \times S \to S$. We denote $a * b$ to mean $*(a,b)$.The structures discussed herein are typically binary operations, but will require several more properties in order to be of use.

Definition 1.2 Let S be a non-empty set and $*$ is a binary operation on S. A group is an ordered pair $(S,*)$ that satisfies the following axioms:

• **Closure axiom**$\forall\, x, y \in S \Rightarrow x * y \in S$

• **Identity:** A binary operation has an identity, denoted e, if $\exists e \in S$ such that $\forall x \in S$, $x*e = x = e*x$

• **Inverse**: $\forall x \in S$ has an inverse if there is an element $x^{-1} \in S$ such that $x*x^{-1} = e = x*x^{-1}$

• **Associativity:** The operation $*$ is associative if $\forall x, y, z \in S$, $(x * y) * z = x * (y * z)$.

Commutativity: The operation $*$ is commutative if $\forall x, y \in S$, $x * y = y * x$.Binary operations do not need to exhibit this property, but it is hugely useful when present. In general, refer to the binary operation of a group as multiplication, unless otherwise stated. Also, to simplify notation, instead of writing $a * b$ for elements in a group, just write ab to denote the multiplication. Additionally, write E to mean $\{e\}$.

Example 1.1 $Q\setminus\{0\}$ is a group under usual multiplication. The identity for this group is 1, and for all $a \in Q\setminus\{0\}$, a^{-1} is$\frac{1}{a}$. Furthermore, $R\setminus\{0\}$ and $C\setminus\{0\}$ are also groups under multiplication, with the same identity and definition of inverses. However, $Z\setminus\{0\}$ is not a group under multiplication, although it is associative and does have an identity, since it lacks the existence of inverses. For example, $4 \in Z\setminus\{0\}$, but $1/4 \notin Z\setminus\{0\}$.

Note$\langle \boldsymbol{R}, * \rangle$, $\langle \boldsymbol{C}, * \rangle$and $\langle \boldsymbol{Q}, * \rangle$ are not groups under multiplication because zero has no inverse.

Definition 1.3 A nonempty subset H of a group G, is a subgroup if H forms a group under the binary operation in G. H is a proper subgroup of G if first it is a subgroup, and second, is a proper subset of G.

If H is a subgroup of G, write$H \leq G$. If H is a proper subgroup of G,write $H < G$.The subgroup $\{e\}$ is called the trivial subgroup of G; a subgroup that is not $\{e\}$ is called a non-trivial subgroup of G.

Example 1.2 S=square root of unit = $\{\pm 1\}$ is $(S,*)$a group under usual multiplication.

Cayley table

Table 2

$*$	1	-1
1	1	-1
-1	-1	1

1. Closure axiom: hold e.g. 1x-1=-1

2. Associative: 1x (1x-1) =-1x (1x1)

3. Identity: 1x1=1, -1x-1=1, e=1

4. Existence of inverse: inverse of 1=1, inverse of -1= -1

5. Commutative property: 1x-1=-1x1

$(S,*)$ is a finite abelian group of order 2

Definition 1.4 Normal Subgroups

A normal subgroup H of G is a subgroup with the additional property that

For all $n \in G$, $nHn^{-1}=H$. In abelian groups, all subgroups trivially have this property, because $nHn^{-1}=nn^{-1}H=H$.

Example 1.3 The alternating group $A_n$of even permutation is a normal subgroup of S_n. Since

(1 2) $\in S_n$ (1 3 2) $\in A_n$ (1 2 3) $\in A_n$ (1 2) (1 2 3) (1 2)$^{-1}$= (1 3 2)

This section introduces finite and infinite cyclic groups, direct products, quotient groups, and order .Then show that the cyclic property is passed down to subgroups. Finally, then prove Lagrange's Theorem and two important corollaries of the theorem.

Definition1.5 A finite cyclic group $G=<x>$ of order m consists of the elements $x^0, x^1, x^2 . . . x^{m-1}$ where $x^{m-1} \neq e = x^m$. The element, x is called a generator of G. An infinite cyclic group is defined $G=<x>=\{x^n : n \in Z\}$ and $|G| = \infty$. Note that if a group operation is additive, define $<x>:=\{nx : n \in Z\}$. Here,

$nx = x + x + . . . + x$, n times.

Example 1.4 A Group G is cyclic ⇔

• $G \cong Z$,if $|G| = \infty$, represented by $(Z, +)$ is an infinite cyclic group generated by 1

• $G \cong Z_m$,if $|G|$equally shown by ($Z_{(m)}$, +) as the set of integers modulo m is a finite cyclic group generated by 1.

Theorem 1.1 A Subgroup of a cyclic group is cyclic.

Proof: If H is a trivial subgroup of G, then $H =< e >$ or $H = G$ is cyclic and we are done. Now suppose H is a nontrivial subgroup of G.

First we take $|G|$ to be finite and cyclic so let b be a generator. Let n be the smallest positive integer such that $b^n \in H$.We wish to prove $H = \langle b^n \rangle$.

Since $b^n \in H$, and H is a group, we have $\langle b^n \rangle \subseteq H$.

Now suppose $h \in H$.Then $h \in G$,and there exists positive integer a such that

$h = b^a$.By the division algorithm, there exist integersq, r such *that* $a = qn + r$ with

$0 \leq r < n$ .Thus $h = b^a = b^{qn+r} = b^{qn}b^r = (b^n)^q b^r . Thus\ b^r = h(b^n)^{-q}$. Note that $h, b^n \in H. As\ b^r \in H and$ $0 \leq r < n$, the minimality condition defining n implies $r = 0$.Thus $h = (b^n)^q b^0 = (b^n)^q e = (b^n)^q \in \langle b^n \rangle$.Hence$H \subseteq \langle b^n \rangle$, and $H = \langle b^n \rangle$ is cyclic. If $\langle b \rangle$ has infinite order, and H is a subgroup $of\ \langle b \rangle$,let a be the least positive integer with $b^a \in H$.If $b^a \in H$ for some $n > a$,we can show a/n by an argument similar to the finite case. Hence,$H = \langle b^a \rangle$. □

Definition 1.6(external) Direct Products

Given groups A and B with operations $(*A)$ and $(*B)$, respectively, the direct product $A \times B$ is the set $\{(a, b)\ such\ that\ a \in A, b \in B\}$, with the operation $*$ defined by $(a_1,b_1) * (a_2,b_2) = (a_1 * a_2,\ b_1 * b_2)$. An important result of this definition is the property that, given groups A and B, their direct product is also a group. The A and B portions of the elements of $A \times B$ do not interact, so the operation preserves the original closure, associativity, and identities of the original groups for each component.

Example 1.5 $(R, +)$ is a group. Therefore,$R \times R = \{(x, y): x, y \in R\}$, with $(a, b) + (c, d) = (a + c, b + d)$, is also a group.

Proposition 1.1 The **Direct product** of two groups is again a group

1) For all $(p, q)(p_1,q_1) \in P \times Q$,we have $p, p_1 \in P$, $q, q_1 \in Q$,so$p, p_1 \in P$ and $q, q_1 \in$Q because P and Q Are closed under multiplication .so $(pp_1,qq_1) \in P \times Q$.Thus a binary operation

On $P \times Q$ or in other words $P \times Q$ is closed under the operation multiplication

2) Associativity in $P \times Q$ Follows clearly from associativity in P and Q.For any $(p, q)(p_1,q_1),(p_2, q_2) \in$

$P \times Q$, Thus

$[(p, q)\ (p_1, q_1)]\ (p_2, q_2)$

$= (pp_1,qq_1)\ (p_2, q_2)$

$= ((pp_1)p_2, (qq_1)q_2)$

$= ((p\ (p_1, p_2), q\ (q_1, q_2)))$

$= (p, q)\ (p_1, p_2,q_1, q_2)$

$= (p, q)\ [(p_1, q_1)\ (p_2, q_2)]$ and so the operation on $P \times Q$ is associative Check for the identity element of $P \times Q$. The only guess which is reasonable would be (1, 1) = $(1_p,1_q)$.indeed, (p, q) (1, 1) = (p, q) for all (p, q)$\in P \times Q$.Thus (1, 1) is the Perfect identity of the product $P \times Q$.

3) Check for the inverse of $(p, q) \in P \times Q$.

Let (p^{-1},q^{-1}) be the inverse. Indeed

$(p, q)\ (p^{-1},q^{-1}) = (pp^{-1}, qq^{-1}) = (1_p, 1_q)\ \forall\ (p, q) \in P \times Q$

So all $(p, q) \in P \times Q$ has the perfect inverse in $P \times Q$ which is $(p, q)^{-1} = (p^{-1},q^{-1})$

Thus $P \times Q$ is a group since it satisfies all the four axioms of the group that is Closure, associative, identity and inverse. □

Definition1.7 If H is a subgroup of a group$G$, and $g \in G$, then the left coset of H in G is the set gH = {gh : h ∈ H}. A right coset of H in G is the set Hg= {hg: h ∈ H}.

Lemma1.1 A subgroup H is a normal subgroup of group G if and only if $gH = Hg$ for all g∈ G.

Proof: Let H be a subgroup of group G. If gH =Hg for all $g \in G$, we have gHg^{-1} = H. Thus $gHg^{-1} \subseteq H$, so H is normal in G. Suppose H is normal in G. Then if $g \in G$, $gHg^{-1} \subseteq H$and $g^{-1}Hg$= $g^{-1}H(g^{-1})^{-1} \subseteq H$. Now, since $gHg^{-1} \subseteq H$, H=$g(g^{-1}Hg)g^{-1}$, out of which we have H = gHg^{-1} or equivalently gH=Hg. □

Definition1.8 If the subgroup H of G is normal, then the sets of left (right) cosets of H in G is itself a group called factor group of G by H (the quotient group of G by H)G/H.

Theorem 1.2 (Holder 1889)

Let G be a group and H be normal subgroup of G .The set $G/H = \{ah \backslash a \in G\}$ is a group operation $(aH)(bH) = abH$

Proof: Let H be a normal subgroup of group G and a, b∈ G.The operation must be independent of the choice of coset representation. Let $aH = a'H$ and $bH = b'H.$ Then $a' = ah_1$ and $b' = bh_2$ for some $h_{1,}h_2$ in H and therefore $a'b'H = ah_1bh_2H = ah_1Hb = aHb = abH$.Note that H is an identity element for G/H. Next, (aH) (bH) = a(bH) H= a(bH) (H) = (ab) (HH) = abH shows the multiplication is well-defined. Associativity holds since ($aHbH$)cH = (abH)cH = $abcH$= aH(bcH) = aH($bHcH$),$eH = H$ is the identity and$a^{-1}H$ is the inverse of aH. □

Example 1.6 Let $G = GLn(R) = \{A \in M_n(R)$: det $(A) \neq 0\}$. Two matrices are in the same coset if and only if they have the same determinant. $N = \{A \in G: det\ (A) = 1\}$ is a normal subgroup of G. Note that $aN = bN$ if and only if $b^{-1}a \in N$.

But $b^{-1}a \in N$ if and only if det $(b^{-1}a)$ = 1 if and only if $det\ (a) = det\ (b)$. Also, $aNbN = abN$ and $det\ (ab) = det\ (a)\ det\ (b)$, so the quotient group G/N is isomorphic to the nonzero real numbers under multiplication. $GLn(R)/SLn(R) \cong R \backslash \{0\}$.

Definition1.9 Let H be a subgroup of a group G. Then the index of H in G is defined to be the number of left cosets of H in G.

Denote the index of H in G by $[G: H]$.

Definition1.10 Order of element

The order of an element g in a group G is the smallest positive integer n such that$g^n = e$ (In addition notation this would be $gn = 0$).If no such integer exist,g has infinite order.The order of an element g is denoted by $O(g)$ or $|g|$

Lemma 1.2 Let G be a group and $H \leq G$.Then every right coset of H in G has the

Same cardinality as H.In particular, any two right cosets have the same cardinality as each other.

Proof: Let Hg be a right coset define $\varphi : H \to Hg$ by $\varphi(h) = hg$

$\varphi$is one to one if $\varphi(h_1) = \varphi(h_2)$, then $h_1g = h_2g$ by right cancellation we have$h_1 = h_2$, for all $h_1, h_2 \in H$ $show\ that$ φ is onto, take $y \in Hg$.Then $y = hg$ for some $h \in H$.In addition, $\varphi(h) = hg = y$. Thus φ is onto. Since φ is one to one and onto, thus a bijection

$|H| = |Hg|$ □

Theorem1.3 (Lagrange's Theorem) If G is a finite group and H is a subgroup of G, then the order of H divides the order of G. Furthermore, the index of H in G equals $|G|/|H|$

Proof: Let $Ha_1, \ldots\ldots Ha_n$ be the right cosets of H in G, then

(By Equivalent classes) let G=$Ha_1 \cup \ldots\ldots \cup Ha_n$ be the disjoint union.

$|G| = |Ha_1 \cup \ldots . Ha_n|$

$= |H| + \ldots . + |H|$

By previous **lemma 1.2**$|H| = |Ha|$

$= n|H|$,So $|H|/|G|$

$n$is the number of distinct right cosets of H in G. □

In general, if n divides$|G|$, then there does not necessarily exist a subgroup H of G such that$|H| = n$. however, in the case of finite abelian groups such subgroups do exist, this will see later.

Corollary1.1 The order of any element of a finite group divides the order of the group.

Proof: Let G be a finite group and let $a \in G$. Then the order of a in G is the same as the order of $<a>$ which divides the order of G. □

Corollary 1.2 Let G be a finite group, and let $a \in G$.Then $a^{|G|} = e$

Proof: Note $o(a) = |\langle a\rangle|/|G|$ by Lagrange's Theorem.Hence $|G| = n.o(a)$ for some $n \in Z$.

Then $a^{|G|} = a^{n.o(a)} = \left(a^{|a|}\right)^n = e^n = e$ □

Corollary 1.3 Group of prime order is cyclic

Proof: Since $|G| = p > 1, G$ has a non-identity element so take any

$x \in G, x \neq e$ Then by Lagrange's Theorem$o(x) = |\langle x\rangle|/|G|$ Since p is prime,$o(x) = 1$ or $o(x) = p$since $x \neq e, o(x) \neq 1$. Hence $o(x) = p$.This means $\langle x\rangle$ has the same number of elements as G, so G=$\langle x\rangle$.

1.3 Isomorphism Theorems

This section defines mappings as homomorphisms and isomorphism. Then also state the isomorphism theorems which will be important later in decomposing abelian groups.

Definition1.11 A **homomorphism**φ from a group G to a group H is a mapping from G into H that preserves the group operation that is for alla, b∈G, $\varphi(ab)$ = $\varphi(a)$ $\varphi(b)$.

Definition1.12 If φ is a homomorphism from $G$to H, then the **kernel** of φ is defined by kerφ = $\{g$∈G: $\varphi(g)$ =$e\}$.

Note that $\varphi(e_G)$ =e_H. However, the subscripts G and H will not be written as the locations of the identities are clear, and in the interests of simpler notation.

Example 1.7 Let $GL(n, R)$ be the multiplicative group of all invertible $n \times n$ matrices, $A, B \in GL(n, R)$ $\det(AB) = detAdetB$ .This means that it is a homomorphism mapping $Gl(n, R)$ into the multiplicativeGroup (R ∗)of non-zero real numbers.

Example 1.8 Let G be a group and N a normal subgroup of G. Then the homomorphism

$\pi: G \rightarrow G/N$

$g \rightarrow gN$

is called the natural homomorphism of G onto G/N. Note that $\pi(g_1, g_2)$ = g_1g_2N = g_1Ng_2N = π (g_1) π (g_2), and $N = ker\ (\pi)$

Proposition 1.2 The complete preimage of a subgroup of G/N is a subgroup of G.

Proof: Let N be a subgroup of group G. Let $\pi: G \rightarrow G/N$ with $g \rightarrow gN$ be the natural homomorphism. Let$g_1g_2N \in G/N$. Then $g_1{g_2}^{-1}N$ = $g_1N{g_2}^{-1}N$ =$g_1N(g_2N)^{-1}$by one step subgroup test, Hence the preimage is closed under multiplication and inverses. □

Definition1.13 An **Isomorphism**φ from a group G to a Group $\bar{G}$ is a one-one mapping (or function) from G onto $\bar{G}$ that preserves the group operation. That is$\varphi(ab) = \varphi(a)\varphi(b) \forall\ a, b \in G$.If there is an isomorphism from G onto $\bar{G}$, it is said that G and $\bar{G}$ are Isomorphic and Write $G \cong \bar{G}$

Theorem1.4 $\boldsymbol{Z}$ is the only infinite cyclic group (up to isomorphism).

Proof: Let $\boldsymbol{G}$ be an infinite cyclic group. Then $\boldsymbol{G}$ =<g>= $\{g^k : k \in Z\}$ for some k∈$\boldsymbol{G}$.

Let $\boldsymbol{\phi}: \boldsymbol{Z} \rightarrow \boldsymbol{G}$

$k \rightarrow g^k$.

1. $\forall x, y \in \boldsymbol{Z}, \phi(x + y) = g(x + y) = g^x * g^y = \boldsymbol{\phi}(x) * (\boldsymbol{\phi} y)$. So $\boldsymbol{\phi}$is a homomorphism
2. $\forall x, y \in \boldsymbol{Z}, \boldsymbol{\phi}(\boldsymbol{x}) = \boldsymbol{\phi}(\boldsymbol{y}) \Leftrightarrow g^x = g^y \Leftrightarrow x = y$ (to within mod k, if G is finite) $\boldsymbol{\phi}$ is one to one
3. $\forall\ y \in \boldsymbol{G}, y = g^k$. Then $\boldsymbol{\phi}(k) = g^k = y$, is $\boldsymbol{\phi}$ onto.

Therefore, $\boldsymbol{\phi}$ is isomorphism and $\boldsymbol{G} \cong \boldsymbol{Z}$. □

Remark: Every cyclic group of infinite order is isomorphic to the additive group Z and every cyclic group of finite order n is isomorphic to the additive group Zn.

2. Structure Theorems for finite Abelian Groups

A group G with binary operation is abelian if, for all $g, h \in G,\ g * h = h * g$. It is customary to use "+" for the binary operation in abelian groups. One of the concepts which reformulate additively and generalize is the concept of "direct product". In abelian Groups it is customary to talk about "direct sum" instead of "direct product". Then note that the direct sum of abelian groups is again abelian. Moreover, call direct sums of infinite cyclic Groups free abelian Groups. Direct sums satisfy an important homomorphism property which gives rise to the important fact that every Abelian group is a homomorphic image of a **free abelian group**. In this section, first make the transition from multiplicative notation to additive notation since our primarily focus is on abelian groups .Then define finite and infinite abelian groups and give some examples .Afterward also alter some definitions (i.e. order in additive notation). Then prove a corollary to Cauchy's Theorem.

Proposition 2.1 Prove that if G is abelian and H is a subgroup of G, then G/H is abelian.

Solution: $(f + H) + (g + H) = (f + g) + H = (g + f) + H = (g + H) + (f + H)$ □

Remark: since cyclic groups are abelian, thus now use additive notation.

Definition 2.1A finite abelian cyclic group $G = \langle a \rangle$ of order m consists of the elements $0, a, 2a, \ldots \ldots, (m - 1)a$.Also, $ma = 0$ and $na \neq 0$ whenever$1 \leq n \leq m - 1$.An infinite abelian cyclic group is defined when

$G = \langle a \rangle = \{na : n \in Z\}$ and $|G| = \infty$

For finite groups, since ma = 0, compute just as with the integers mod m; thus finite G of order m is isomorphic to the additive group Z_m of residue classes of the integers mod m. All finite cyclic groups of the same order m are isomorphic, so denote them all as Z_m.

Proposition 2.2 Cyclic groups are abelian

Proof: Let G=$\langle a\rangle$.If $g_1, g_2 \in G$,then $g_1 = ma\ and\ g_2 = na$ for some $m, n \in Z$.

Then $$\begin{aligned} g_1 + g_2 &= ma + na \\ &= (m+n)a \\ &= (n+m)a \\ &= na + ma \\ &= g_2 + g_1 \end{aligned}$$ □

Definition 2.2 for G an Abelian group, and $g \in G$, the order of g in G is the smallest $n \in N$ such that ng = e. If there is no such $n \in N$, we say g is of infinite order.We will denote the order of g as $o(g)$.

Definition2.3 A p-group is a group, in which the orders of all the elements consist only of powers of a fixed prime, p

Definition2.4 For a finite group G and a prime number p, a subgroup p of G is called a Sylow p-subgroup of G if |p| = p^{α} for some integer $\alpha \geq 1$ such that p^{α} is a divisor of $|G|$ but $p^{\alpha+1}$ is not.

Corollary 2.1 (Corollary to Cauchy's Theorem) A finite group,G is a p-group if and only if $|G|$ = p^n for some prime p, and some $n \in N$.

Proof: Let G be a finite p-group, so that the elements of G have order of the form p^n for some fixed prime p, and some $n \in N$. Cauchy's Theorem implies that all subgroups of G must have orders of the form p^n. Thus, $|G|$ =p^n. Conversely, if$|G|$ = p^n for some prime p, and some $n \in N$, then by Lagrange's Theorem, the order of every element in G must divide p^n. Thus the order of every element in G must be of the form p^i for some $i \in N$. Therefore, G is a p-group. □

Theorem 2.1 The Fundamental Theorem of Finite abelian Groups. Every finite Abelian group G is the direct sum of cyclic groups, each of prime power order.

Example 2.1 Find all abelian groups, up to isomorphism, of order 56

We first split G into p-groups. $|G|$ = 56 = $2^3.7$

$G \cong G_{2^3} \oplus G_7$

For G_{2^3}, look at the exponent.

3 $Z_{2^3} \oplus Z_{7^1} = Z_8 \oplus Z_7$

2, 1 $Z_{2^2} \oplus Z_{2^1} \oplus Z_{7^1} = Z_4 \oplus Z_2 \oplus Z_7$

1, 1, 1 $Z_{2^1} \oplus Z_{2^1} \oplus Z_{2^1} \oplus Z_{7^1} = Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_7$

For G_7, there is only one way to express the group. Thus, G is isomorphic to one of the following

$Z_8 \oplus Z_7$

$Z_4 \oplus Z_2 \oplus Z_7$

$Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_7$

Theorem2.2 (Invariant Factor Decomposition) Let G be a finite abelian group. Then G is isomorphic to a direct sum of cyclic groups $Z_{n_1} \oplus Z_{n_2} \oplus \cdots \oplus Z_{n_k}$ such that $n_i | n_{i-1}$ for all

i = 2,3... k. Furthermore, the n_i are uniquely determined by G.

Example 2.2 Suppose G = $Z_8 \oplus Z_8 \oplus Z_4 \oplus Z_2 \oplus Z_2 \oplus Z_{27} \oplus Z_3 \oplus Z_3 \oplus Z_{11} \oplus Z_{11}$

Write out these prime powers in matrix format, one row for each prime base, left-justified, ordered from the least exponent to the greatest, as follows.

2	2	4	8	8
		3	3	27
			11	11
2	2	12	264	2376

Observe that every column consists of distinct prime powers. As you probably have guessed, then conclude that

$n_1 = 8 . 27 . 11 = 2376$

$n_2 = 8 . 3 . 11 = 264$

$n_3 = 4 . 3 = 12$

$$n_4 = 2 = n_5$$

This consists of successive divisors—each number divides the preceding number.

That is G = $Z_2 \oplus Z_2 \oplus Z_{12} \oplus Z_{264} \oplus Z_{2376}$ with 2|2|12|264|2376

3. Torsion and Torsion-free Groups

This section looks at infinite abelian groups and define when a group is torsion, torsion-free, and mixed. Consider the following three examples of abelian groups:Q the additive group of rationals; Q/Z the factor group of the additive group of the rationals by the integers; ℂ, the multiplicative group of complex numbers. Each of these groups is not isomorphic to the others, but how would we prove that? One way is to examine the orders of the elements of the groups. Now every element of Q except 0 is of infinite order and every element of Q/Z is of finite order. For if $r \in Q$, r = m/n where m, n are two integers. Thus m$(r + Z) = mr + Z = n + Z = Z$. Let us, to avoid confusion, continue to use the multiplicative notation for ℂ. We assert that ℂ has elements of infinite order and also elements of finite order. Recall that the identity of ℂ is 1. Note that $(-1)^3$ = 1 implies that -1 is of order 3 and 4^r = 1 if and only if r = 0. Hence -1 is of finite order and 4 is of infinite order. Summarizing, we have

(i) Q has every element but the identity of infinite order.
(ii) Q/Z has every element of finite order.
(iii) ℂ have elements of finite order and elements of infinite order.

Definition3.1 A **torsion group** is a group in which every element is of finite order.

Example 3.1 All finite groups are torsion groups. If G is a finite group and x∈G, then

O (x) divides O (G) < ∞. Thus G is a torsion group.

Example 3.2 G =$\prod_{k=1}^{\infty} Z_p = Z_p \times Z_p \times. . .$

is a torsion group. Let x∈G. Then $px = e$, hence O (x)≤ p. So, G is a torsion group expressed as an infinite direct product.

Theorem 3.1 Show that the class of torsion abelian groups is closed under direct sums, quotients and subgroups.

Proof. Let H be an abelian group. Assume $h \in T(\bigoplus_{\alpha \in j} H_\alpha)$ .Then there exists a finite n Suchthat $nh = \bigoplus E_\alpha$ .Therefore $h \in \bigoplus T(H_\alpha)$. Assume $h \in \bigoplus T(H_\alpha)$, with o $(h_i) = n_i$ in H_i.Then $\prod_{all\ i} n_i$ is finite, which means h has finite order in $\bigoplus H_\alpha$.Thus, torsion groups are closed under direct sums.

Let $a \in T, o(a) = n$ in T and $(a + N) \in T/N$.Then $n(a + N) = na + nN = e + N = N$,hence the order of $a + N$ is a divisor of n in T/N. Thus, torsion groups are closed under quotients. Let T be a torsion abelian group,H a subgroup of $T$and $h \in H$.Then the order of h in H divides the order of $h$in T,which is finite. Thus, the order of h in H is finite, so H must also be torsion, proving closure of subgroups. □ However, the torsion class is not closed under direct products, which will be demonstrated in this example.

Example 3.2 Let G=$Z_2 \times Z_3 \times Z_5 \times Z_7 \times ...$ .be a set of all primes. Then the order of set G which is the smallest positive integer will be infinite, since no finite $n \neq 0$ exists such that $n(1,1,1, . . .) = e$

Definition 3.2 An abelian group$G$, is **torsion-free** if all its elements, except for the identity, have infinite order. The **torsion part** of G is the set of all elements in G of finite order, denoted by $T\ (G)$.

Example 3.3 Both Q and Z have an infinite number of element and are classified as infinity Abelian group. It is clear that $n = 0$ if and only if $nx = 0$.Therefore, Q and Z are called **torsion free.**

Example 3.4 The torsion part of $R/Z$is Q/Z.

Solution: Suppose $r + Z$ is of finite order (r ∈ R). Then for some nonzero

integer $n, n(r + Z) = Z$. But $n(r + Z) = nr + Z$, and so $nr \in Z$. This means that r is a rational number. Thus $T(R/Z) \subseteq Q/Z$, where Q is the subgroup of rational numbers. On the other hand, if $a + Z \in Q/Z$, then $a = m/n$ where $m, n \in Z$ and n$\neq O$. So $n\,(a + Z) = n(m/n + Z) = n(m/n) + Z = m + Z = Z$ Hence $a + Z$ is of finite order and $Q/Z \subseteq T(R/Z)$. Thus we have proved that $Q/Z = T(R/Z)$.

Definition3.3 An abelian group is called mixed when it contains elements of both finite and infinite order.

Example 3.5 $Z_3 \bigoplus Z$ is a mixed group. This is clear since Z_3 is a torsion group and Z is torsion-free

. We have O (1, 0) = 3 and O (0, 1) = ∞.

Example 3.6 Given G=$Z_{p^3} \times Z_{p^6} \times Z_{p^9} \times Z_{p^{12}} \times. \ . \ . \ .$ is a mixed group.

In this group we can see that O(1,0,0,0,...) = p and O(1,1,1,...) = ∞.

Theorem3.2 $T\ (G)$ is a subgroup of G an abelian group (termed the torsion subgroup of G).

Proof; clearly,$T(G) \neq \emptyset$ as $e \in T(G)$

Let $a, b \in G$ be of order m, n∈N such that $am = 0$ and $bn = 0$respectively. Thus, $mn(\ a - b)$ = $mna - mnb$ = 0 - 0 = 0

Thus if $a, b \in T\ (G)$, $a - b \in\ T(G)$ and $T\ (G)$ is a subgroup of G. □

Theorem3.3 For an abelian group, G, the quotient group $G/T\ (G)$ is torsion-free.

Proof: Now consider $G/T\ (G)$. Assume g + T (G) is of finite order n, i.e.

n $(g + T\ (G))$ = ng+ T (G) = $T\ (G)$. It follows that n$g \in T(G)$. As $T(G)$ consists of all the elements of G of finite order, there exists m such that m (ng) = O.

Then g is of finite order and $g \in T\ (G)$; hence $g + T(G)\ =\ T(G)$. Therefore, the only element of finite order in $G/T(G)$ is the zero $T(G)$. Thus $G/T(G)$ is torsion-free. □

Example 3.7 R/Q is torsion-free since $nx \in Q$ if and only if $x \in\ Q$.

Let x ϵ R, then $x + Q \in\ R/Q$.Since the only finite element is the identity E, then

$n(x + Q) = nx + Q$=$E + Q\ = Q$ and happens only when $x \in Q$.

Theorem 3.4 The class of torsion-free abelian groups is closed under subgroups and direct products.

Proof: Let F be a torsion-free abelian group, H a subgroup of F and $e \neq h\ \in\ H$. Suppose the order of h is finite in H. Then it must divide the order of h in F implying h is a nontrivial torsion element of F giving a contradiction. Thus the order of h must be infinite in H, meaning H must also be torsion-free.

Let F_1,F_2,F_3,... be torsion-free abelian groups and define F to be the direct product of all F_i. Let e$\neq x$ = (x_1,x_2,x_3,...)∈ F with $O(x_i)$ = ∞ in F_i. Suppose O(x) = n in F. Then there exists nontrivialx_j∈$F_j$such that

$O(x_j)$divides n in $F_j$and hence is finite which contradicts F_j being torsion-free. Therefore, O(x) = ∞ in F, meaning F must also be torsion-free. □

Remark: As it has been shown that $T(R/Z)\ =\ Q/Z$. Thus by **Theorem 3.3**, $R/Q \cong\ (R/Z)/\ (Q/Z)$ is torsion-free.

Definition 3.4 A **torsion group** is a group in which every element is of finite order.

Example 3.8 All finite groups are torsion groups. If G is a finite group and x∈G, then

$O\ (x)$ divides $O\ (G) < \infty$. Thus G is a torsion group.

Example 3.2 G =$\prod_{k=1}^{\infty} Z_p = Z_p \times Z_p \times. . .$

is a torsion group. Let x∈G. Then $px = e$, hence $O\ (x)$≤ p. So, G is a torsion group expressed as an infinite direct product.

Theorem 3.5 Show that the class of torsion abelian groups is closed under direct sums, quotients and subgroups.

Proof. Let H be an abelian group. Assume $h \in T(\bigoplus_{\alpha\in j} H_\alpha)$ .Then there exists a finite n Suchthat $nh =$

$\bigoplus E_\alpha$.Therefore $h \in\bigoplus T(H_\alpha)$.

Assume $h \in\bigoplus T(H_\alpha)$, with o $(h_i) = n_i$ in H_i.Then $\prod_{all\ i} n_i$ is finite, which means h has finite order in $\bigoplus H_\alpha$.Thus, torsion groups are closed under direct sums.

Let $a \in T, o(a) = n$ in T and $(a + N) \in T/N$.Then

$n(a + N) = na + nN = e + N = N$,hence the order of $a + N$ is a divisor of n in T/N. Thus, torsion groups are closed under quotients.

Let T be a torsion abelian group,H a subgroup of $T$and $h \in H$.Then the order of h in H divides the order of h in T,which is finite. Thus, the order of h in H is finite, so H must also be torsion, proving closure of subgroups. □

However, the torsion class is not closed under direct products, which will be demonstrated in this example.

Example 3.9 Let G=$Z_2 \times Z_3 \times Z_5 \times Z_7 \times. . .$.be a set of all primes. Then the order of set G which is the smallest positive integer will be infinite, since no finite $n \neq 0$ exists such that $n(1,1,1, . . .) = e$

Definition 3.5 An Abelian group G is **torsion-free** if all its elements, except for the identity, have infinite order. The **torsion part** of G is the set of all elements in G of finite order, denoted by $T\ (G)$.

Example 3.10 Both Q and Z have an infinite number of element and are classified as infinity Abelian group. It is clear that $n = 0$ if and only if $nx = 0$.Therefore, Q and Z are called **torsion free.**

Example 3.11 The torsion part of R/Z isQ/Z.

Solution: Suppose $r + Z$ is of finite order (r ∈ R). Then for some nonzero integer $n, n(r + Z) = Z$. But $n(r + Z) = nr + Z$, and so $nr \in Z$. This means that r is a rational number. Thus $T(R/Z) \subseteq Q/Z$, where Q is the subgroup of rational numbers. On the other hand, if $a + Z \in Q/Z$, then $a = m/n$ where $m, n \in Z$ and n$\neq O$. So $n\,(a + Z) = n(m/n + Z) = n(m/n) + Z = m + Z = Z$ Hence $a + Z$ is of finite order and $Q/Z \subseteq T(R/Z)$. Thus, we have proved that $Q/Z = T(R/Z)$.

Definition3.6 An abelian group is called mixed when it contains elements of both finite and infinite order.

Example 3.12 $Z_3 \bigoplus Z$ is a mixed group. This is clear since Z_3 is a torsion group and Z is torsion-free.

We have O(1, 0) = 3 and O(0, 1) = ∞.

Example 3.6 Given G=$Z_{p^3} \times Z_{p^6} \times Z_{p^9} \times Z_{p^{12}} \times. . . .$ is a mixed group.

In this group we can see that O(1,0,0,0,...) = p and O(1,1,1,...) = ∞.

Theorem 3.6 $T\ (G)$ is a subgroup of G an abelian group (termed the torsion subgroup of G).

Proof; clearly,$T(G) \neq \emptyset$ as $e \in T(G)$

Let $a, b \in G$ be of order m, n ∈ N such that $am = 0$ and $bn = 0$respectively. Thus, $mn(\ a - b)$ = $mna - mnb$ = 0 - 0 = 0

Thus if $a, b \in T\ (G)$, $a - b \in\ T(G)$ and $T\ (G)$ is a subgroup of G. □

Theorem3.3 For an abelian group, G the quotient group $G/T\ (G)$ is torsion-free.

Proof: Now consider $G/T\ (G)$. Assume g + T (G) is of finite order n, i.e.

n (g + $T\ (G)$) = ng+ T (G) = $T\ (G)$. It follows that n$g \in T(G)$. As $T(G)$ consists of all the elements of G of finite order, there exists m such that m (ng) = O.

Then g is of finite order and $g \in T\ (G)$; hence g + $T(G)\ =\ T(G)$. Therefore, the only element of finite order in $G/T(G)$ is the zero $T(G)$. Thus $G/T(G)$ is torsion-free. □

Example 3.13 R/Q is torsion-free since $nx \in Q$ if and only if $x \in\ Q$.

Let x ϵ R, then $x + Q \in\ R/Q$.Since the only finite element is the identity E, then

$n(x + Q) = nx + Q$=$E + Q = Q$ and happens only when $x \in Q$.

Remark: As it has been shown that$T(R/Z)\ =\ Q/Z$.Thus by **Theorem 3.3**, $R/Q \cong\ (R/Z)/\ (Q/Z)$ is torsion-free.

This section investigates finitely generated abelian groups with definitions and examples.

Definition 3.6 An abelian group G is finitelygenerated if there exist finitely many elements,$x_{1,} x_2 ... x_s \in G$ such that every $x \in G$ can be written in the form$x = n_1 x_1 + \cdots + n_s x_s$, for some $n_i \in Z$. In this case, we write G =<x_1,$x_{2.}$...x_s>.

In this case, we say that the set $\{x_1 . . . . . x_s\}$ is a generating set of G or that $x_1\ .\ .\ .\ .\ x_s$ generate G.

A **basis** of G is a linearly independent subset of G which also generates G. If G is finitely generated, then the minimal s for which G =<x_1,x_2...x_s> is called the rank of G. Note that rank(G) $=$ 1 if and only if G is cyclic.

Example 3.13 Z and Z_n are finitely generated. Since each is cyclic,Z =< 1 > and Z_n=< 1 >.show that $Z =< 1 >$ since by definition$< 1 >= \{n. 1\ /n \in Z\}$ (The cyclic group generated by 1)

Example 3.14 Q is not finitely generated.

Assume Q is finitely generated. Then Q is cyclic and has a generator $\frac{p}{q}$ where$p, q \in Z$. Show that

$\frac{p}{q+1} \notin \langle\frac{p}{q}\rangle$. If $\frac{p}{q} \epsilon \langle\frac{p}{q}\rangle$ then there exist$z \in Z$ such that $\frac{p}{q} = \frac{p}{q+1}$. Then, $z = \frac{p}{q} . \frac{p}{q+1} = \frac{q}{q+1}$. The latter is irreducible and implies that$z \notin$ Z. Thus, arrive at a contradiction

Then show that divisibility is closed under subgroups, quotients, sums and products. To show that a divisible abelian group is the direct sum of its torsion subgroup and its torsion-free quotient and further analyze each summand.

Definition 3.7 A group G is said to be divisible if for each integer $n \neq 0$ and each element $g \epsilon G$ there exists $h \in G$ such that $nh = g$.

Alternatively, G is divisible if $G = nG$ for every integer n.**Note** that a cyclic group is not divisible. Nor for that matter is a direct sum of cyclic group.

Example 3.15 The addition group of rational numbers, denoted by Q, is divisible. Given any rational a and any integer $n > 0$, there exists $a' = \frac{a}{n} \in$ Q such that $a = a'n$.Also the following groups are divisible: the additive group of real numbers, the additive group of complex and the multiplicative group of real numbers.

Example 3.16 Z is not divisible. Take $g = 1$ and $n = 5$ to see if $5h = 1$, then $h = 1/5 \notin Z$. This example demonstrates that divisibility is not closed under subgroups since Z is a subgroup of Q.

Definition 3.8 A trivial abelian group A is called divisible if for each element $a \in A$ and each nonzero integer K, there is an element $x \in A$ such that $x^k = a$. (Here the group operation of A is written multiplicatively. In addition notation, the equation is written as $kx = a$)that is, A is divisible if each element has a k-th root in A.

Example 3.17 Nontrivial finite abelian groups are not divisible. Let G_n be a finite abelian group of order$n$and $e \neq g \in G_n$. Then for every$h \in G_n$, we have $e = nh \neq g$.

Example 3.18 The group $Q/_Z$ is a torsion and divisible.Let $g \in Q$ and $n \in N$. Just as before, define $h = (1/n)g \in Q$. Then$n\ h + Z = n\,((1/n)g + Z) = g + Z = nh + nZ = n\,(h + Z)$.

Theorem 3.7 The class of divisible groups is closed under quotients

Proof: Let G be a divisible group, N a subgroup of $G$and $x \in G$. Then for every $n \in Z$, there exists $y \in G$ such that $ny = x$. Then $n(y + N) = ny + nN = x + N$. Thus the same$y$ can be taken to show that G/N is also divisible proving closure under quotients. □

Theorem 3.8 The direct sum of abelian groups is divisible if and only if each summand is divisible.

Proof: Let $H\gamma, \gamma \in \Gamma$ be abelian groups and let $H = \oplus H\gamma$. Suppose Hγ is divisible for all γ ∈ Γ. Assume $h =$

$(h\gamma) \in H$ and n is a positive integer. Then there exists $t = t\gamma \in H\gamma$ such that $nt\gamma = h\gamma$, which shows that G is divisible.

Conversely, suppose H is divisible, $h_{\gamma_o} \in H\gamma_o$ and n is a positive integer. Let $h = (h\gamma) \in H$ where $h\gamma$ = e if $\gamma \neq \gamma_o$ and $h_{\gamma_o} = h_{\gamma_o}$. Since H is divisible, there exists t = (tγ) $\in H$ such that $nt = h$. But then, n$t_{\gamma_o} = h_{\gamma_o}$, which shows $H\gamma_o$ is divisible. □

Definition 3.9 An Abelian group is called free-abelian if it is a direct sum of infinite cyclic groups.

Definition 3.10 A subset $\{e_i\}_{i\in I}$ of an abelian group A is called a basis for A if $\forall x \in A$ there exists a unique representation $x = \sum_{i\in I} x_i e_i$ with $x_i \in Z$ and almost all $x_i = 0$,an abelian group is called free if it has a basis.

Example 3.19$Z_m = Z/mZ$ is not free because the representation is not unique, which means it cannot be expressed as a linear combination i.e. it has got no basis.

Example 3.20 $Z \oplus Z$ is a free abelian group with basis $\{e_1 = (1,0), e_2 = (0,1)\}$

The results are shown as an ***Application*** below:

A general theory of infinite-dimensional Lie groups is hardly developed. Even Bourbaki only develops a theory of infinite-dimensional manifolds, but all of the important theorems about Lie groups are stated for finite-dimensional ones. An infinite-dimensional Lie group G is a group and has an infinite-dimensional manifold with smooth group operations

$$m: G \times G \to G \quad , m\,(g,h) = g \cdot h \quad C^\infty,$$

$$i: G \to G, \quad i\,(g) = g^i, C^\infty.$$

Such a Lie group G is locally diffeomorphic to an infinite-dimensional vector space. This can be a Banach space whose topology is given by a norm$\|\cdot\|$, a Hilbert space whose topology is given by an inner product $<\cdot,\cdot>$, or a Frechet space whose topology is given by a metric but not by a norm. Depending on the choice of the topology on G, Banach, Hilbert, or Frechet Lie groups, respectively. The Lie algebra g of a Lie group G is defined as $g =${Left invariant vector fields on $G \simeq T_eG$ (tangentspaceattheidentity e).The isomorphism is given (as infinite dimensions)by

$$\xi \in T_eG \rightarrow X_\xi \in g, \qquad X_\xi(g) : T_eL_g(\xi),$$

And the Lie bracket on g is induced by the Lie bracket of left invariant vector fields

$$[\xi,\eta] = [X_\xi, X_\eta](e), \xi, \eta \in T_eG.$$

The definition in infinite dimension is identical with the definition in finite dimensions. The big difference although is that infinite-dimensional manifolds its Lie groups are not locally compact. Some classical examples of finite-dimensional Lie groups are the matrix groups $GL(n), SL(n), O(n), SO(n), U(n), SU(n), and\ Sp(n)$with smooth group operations given by matrix multiplication and matrixinversion. The Lie algebra bracket is the commutator

$[A, B] = AB - BA$ with exponential map given by

$exp\ (A) = \sum_{t=0}^{\infty}\left(\frac{1}{t!}\right)A^t = e^A$.However, only the U (1) and $S0$ (2) are **abelian groups and** this will be shown below:

A rotation in two dimensions is an example of an **infinite abelian group.**

In two dimensions' rotations are uniquely defined by the angle of rotation. They preserve the length of a vector and the angle between vectors. Two successive rotations is a rotation, the rotation by $\theta = 0$ is the identity, and any rotation can be undone by rotating in the opposite direction. The set of all two-dimensional rotations forms a group, called $\boldsymbol{U}\ (\mathbf{1})$.The elements of the group are labelled by the angle of the rotational $\theta \in [0, \pi)$. There are infinite number of elements, denoted by a continuous parameter; groups where the elements are labelled by continuous parameters are called continuous groups. Then denote two–dimensional rotations by $R(\theta)$. Note that the parameter labeling the rotations varies in a compact interval (the interval $[0,2\pi)$ in this case). Groups with parameters varying over compact intervals are called compact groups. The action of rotations on real vectors in two dimensions known as $\boldsymbol{SO}(\mathbf{2})$defines a representation of the group. Intuitively two successive rotations by $\Theta + \psi$yield a rotation by $\Theta + \psi$, and hence the group of two-dimensional rotations is abelians. It is interesting to consider a one dimensional complex representation of $\boldsymbol{U}\ (\mathbf{1})$.Given that the coordinates (x, y) of a point in a two-dimensional space, define the complex number $Z = x + iy$. The transformation properties of Z define a representation $Z \longmapsto Z' = e^{i\theta}Z$

Knowing that $e^{i\Theta}$=cosθ+isinθ

$x + iy \to x' + y' = (cos\theta\ + isin\theta)\ (x + iy)$

$= cos\theta x - \ sin\theta y\ +\ i\ (cos\theta y\ + sin\theta x)$ which is

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} cos\Theta & -sin\Theta \\ sin\Theta & cos\Theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

U (1)$\cong$ $\boldsymbol{SO}(\mathbf{2})$ and it is an infinite abelian group.

Application in algebraic Topology

It is often the case that scientific observations yield a set of n-dimensional points used to represent a space. Examples include coordinates of locations on the globe identified by GPS (Global Positioning System), coordinates of atoms in a protein measured by X-ray Crystallography, or coordinates of bodies in space. In any case, we would like to take that point set, and describe the shape of the space it represents. This type of analysis lends itself naturally to a homology theory called **simplicial homology**. Simplicial homology is a classical tool of topology, developed by **Poincare** (Analysis Situs, 1895) (8), which takes a simplicial complex and yields the homology groups of the underlying space. To understand exactly what this means, begin with the following **definitions:**

Definition 1: A simplicial complex is called directed if each simplex of dimension greater than zero has an associated direction. For example, the direction of an edge can be defined by determining a starting and terminating endpoint. Each face, or 2-simplex, can have a clockwise or counterclockwise direction. Direction of higher dimensional simplices can be defined similarly by providing an order of vertices. Then, would like to be able to move through a simplicial complex, in order to study its structure, keeping track of simplices and direction as we move. To do this, define an algebraic structure called a **chain**.

Definition 2: Let Kbe a directed m-complex.A K-chain C is defined as

$$\sum_{i=1}^{m} a_i \delta_i$$

Where $a_i \in \delta_i \in K$.The set of all K-chains in a complex, called a chain complex, is denoted by c_k for $k = 0,1,2. . .$ for completeness, define $0\delta = \emptyset$,the zero.

Definition 3: Let K be a directed complex and δ a K-simplex in K.The boundary of δ,denoted by $\partial(\delta)$,is the K-chain defined as the sum of the $(K-1)$ faces of $\delta$with the signs reflecting the orientation of the faces

Remark 1 Therefore, the boundary operator maps a k-simplex to a chain of (k-1) - simplices

Definition 4 Let K be a directed complex and C be a K-chain in K defined by

$C = a_1\,\delta_1 + a_2\delta_2 + . . . + a_n\delta_n$, where δ_i are simplices, then the boundary operator

$\partial: C_k \to C_{k-1}$(for $K > 0$) applied to C is defined as $\partial(C) = a_1\,\partial(\delta_1) + a_2\partial(\delta_2) + . . . + a_n\partial(\delta_n)$

Definition 4 Let Cbe a k-cycle in the directed complex$K$. If there exists a $(k+1)$ −chain D such that $\partial(D) = C$, then C is called a k-boundary. The set of all k-boundaries is denotedB_k. Also describe B_k as the image of:$C_{k+1}(K) \to C_k(K)$.

Since the group is an algebraic structure, talk about sub-groups, **kernels**, **images**, etc

By definition 4 the kernel of $\partial(C)$ for $c \in C_k$ is the set of chains C such that $\partial(c)$ = 0.Then denote this set$Z_k \subset C_k$, and calls the elements of $Z_k K$-cycles**.**

Remark 2: image$B_k \subseteq Z_k \subseteq C_k$.The k-th homology group is the quotient of the cycle group over the boundary group:

$$H_k = {}^{Z_k}/_{B_k}$$

Homology groups are always abelian. Furthermore, for spaces that are considered finitely generated.

Thus a topological space X with basic sequence of groups called chains with C_0 (0-dimensional chain),C_1(1-dimensional chain),and C_2(2-dimensional chains) Then between the chains, there are maps called the boundary maps displayed as follows:,∂_2,∂_1 and ∂_0.Then the,2- dimensional chains shows little discs,1-dimensional chain shows edges and 0-dimensional chain which is the vertices. **Note**: There are two ways to define the boundary operator ∂_0 on 0-chains. The first approach is to make $\partial_0(C_0) = 0$ for all 0-chains. With this definition, the dimension of the resulting homology group counts the number of path-connected components of a space.

$$\begin{array}{ccccccc}
C_2 & \xrightarrow{\partial_2} & C_1 & \xrightarrow{\partial_1} & C_0 & \xrightarrow{\partial_0} & 0 \\
0 & & Z \oplus Z \oplus Z & & Z \oplus Z \oplus Z & & \\
0 & & a \quad b \quad c & & x \quad y \quad z & & \\
 & & la + mb + nc & & \alpha x + \beta y + \gamma Z & & \\
 & & 2^{nd}\text{ dim} & & 1^{st}\text{ dim} & &
\end{array}$$

C_2 is zero considering the skeleton of the triangle, C_1 is a linear combination of edges of a triangle i.e.$\{a, b, c\}$ whilst C_0 is a linear combination of vertices$\{x, y, z\}$.Then below is the operation of each boundary $\partial_{0,}\partial_1, \partial_2$acting on each respective dimensional chain$C_{0\,,}C_1 and\ C_2$.

$$\begin{aligned}
\partial_0 \ &: \quad x, y, z \rightarrow 0 \\
\partial_1 \ &: \quad a \rightarrow y - x \\
&\quad\ \ b \rightarrow z - y \\
&\quad\ \ c \rightarrow x - z \\
\eth_2 \ &: \quad 0 \rightarrow 0
\end{aligned}$$

Let us look at the 0-dimensional chain.

Z_0 =Kerð$_0 = C_0 =< x, y, z >$

B_0 =Im$\partial_1 =< y - x, z - y, x - z >$

Homology of 0-dimensional chain is $H_0 = {Z_0}/{B_0} = {< x, y, z >}/{< y - x, z - y, x - z >}$

Quotient by B_0 =setting elements of B_0 to 0

$y - x = 0$ *implies* $y = x$

$z - y$ =0 *means* $z = y$

Which implies that $x = y = z$.

$H_0 \simeq Z$ This is a copy of infinite cyclic group, which is the simplest infinite abelian group and be our element of the form $nx + B_0$ or $H_0 \cong {Z \oplus Z \oplus Z}/{Z \oplus Z} \cong$ Z.Then also look at the 1-dimensional chain.

$$H_1 \equiv {Z_1}/{B_1}$$

Z_1 =Ker ð$_1$

$\partial\ (la + mb + nc) = l(y - x) + m(z - y) + n(x - z)$

$$=(n - l)x + (l - m)y + (m - n)z = 0 \implies l = m = n$$

So Ker $\partial_1 = < a + b + c >= Z_1 \simeq Z$ (groups of cycles)

Since C_2 is 0 meaning that $\partial_2 : 0 \longrightarrow 0$

$$B_1 = \text{imð}_2 = 0$$

$$H_1 \equiv {Z_1}/{B_1}$$

$$\equiv {Z_1}/{0}$$

$$\equiv Z_1 \simeq \text{Z}$$

The Homology of 0-dimensional chain is $H_0(S^1)$ = Z

The homology of 1-dimensional chain is $H_1(S^1) = Z$

Finally, the homology of n-dimensional chain will be $H_n(S^1) = 0 \quad n \geq 2$

H_n(X) measures the number of connected components.

Thus Z is a simplest infinite abelian group which is homeomorphic to a circle as shown above in the computational of the homology of a skeleton triangle with 0-dimensional and 1-dimensional chain

4. Results and Discussion

The Theory of abelian groups is generally simpler than that of their non-abelian counterparts and finite abelian groups are very well understood. Classifying groups by stating whether elements are finite or infinite and categories them as infinite abelian Groups of Torsion, Torsion Free and mixed group. Every torsion group splits into a direct sum of primary groups and this decomposition is unique. A free abelian group is a direct sum of infinite cyclic groups. Direct sums of infinite cyclic groups are all the free abelian groups. An important fact is that every Abelian group is a homomorphic image of a free abelian group. The torsion group $T(G)$ of a group G was defined and it was shown that $G/T(G)$ is torsion-free. Every subgroup of a free abelian group is free abelian. All divisible abelian groups turn out to be direct sums of groups isomorphic to Q and the groups, and the cardinalities of the sets of components isomorphic to Q, as well as to $Z_p\infty$ (for each), from a complete and independent system of invariants of the divisible group. For example, every theorem that is stated in this paper may be generalized for application in Groups and Ring Theorem and also in algebraic Topology. In **conclusion** two important special classes of infinite abelian groups with diametrically opposite properties are torsion groups and torsion-free groups, exemplified by the groups $^{\mathbb{Q}}/_{\mathbb{Z}}$ (periodic) and $\mathbb{Z}$ (torsion free).in addition, many large abelian groups possess a natural **topology,** which turns them into topological groups. Several classes of torsion-free abelian groups have been studied extensively e.g. Free abelian groups. Amongst torsion-free abelian groups of finite rank, only the finitely generated case and the rank 1 are well understood. lastly Finite abelian groups remain a topic of research in **computational group theory.**

5. Recommendation

the theory of infinite abelian groups is an area of current research and basically the structure theorem for infinite abelian groups depends on group analysis of the set of integers (Z),integers modulo (Z_n) and also an additive of rationale (Q).However, more discovery is still on for describing the structure theorem of an infinite abelian group.

Acknowledgements

I would like to thank Dr B. Chilambwe and My Family for their time, energy and continued support over the years of my research process.

Reference

[1]. Kaplausky, I.1968.Infinite abelian groups. University of Michigan Press, Ann Arbor, Michigan.89p.

[2]. J.B Fraleigh, A First Course in Abstract Algebra, 7th edition, Addison Wesley, 2003

[3]. I.N.Herstein, Topics in Algebra, 2nd edition, John Wiley& Sons 1975

[4]. Joseph J. Rotman, An Introduction to the Theory of Groups. 4th Edition, New York, New York, Springer-Verlag, 1995.

[5]. Bing, some aspects of the topology of 3-manifolds related to the Poincare conjecture. In lectures On modern mathematics, Vol. (ii), T.L Scaty(Ed), New York, 93-128

[6]. Rotman, Joseph J.1965.The theory of Groups: an Introduction, Allyn and Bacon, Inc, Boston, Massachusetts. 304p

[7]. L. Fuchs, "Infinite abelian groups”, 2, Acad. Press (1973)

[8]. Thomas W. Hungerford, Algebra. New York, New York Holt, Rinehart and Winston, 1974.

[9]. David S. Dummit, Richard M. Foote, Abstract Algebra. 2nd Edition, New York, New York, John Wiley and Sons, 1999.

[10]. Bredon, G. (1993). Topology & Geometry. New York: Springer-Verlag New York, Inc.

[11]. W. Keith Nicholson, Introduction to Abstract Algebra. 2nd Edition New York, New York, John Wiley and Sons, 1999.

[12]. P.Garret,Website:http://WWW.umn.edu/garret/m/algebra/,last modified 2013.

[13]. B.L Van der Waerden, A history of algebra, springer-Verlag, Berlin, 1980.149p