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Calculation of Error between the Exact Solution and Solution of Parabolic Equation (Heat Equation) by Krylov Approximation Methods.

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Abstract

Having good estimates in the computation of the approximation to expressions for the form $f(A)v$ is very important in practical applications if we know at what stage the algorithm has to stop i.e avoid the principle of "luckybreak". In this paper we develop an a posteriori upper bound on the Krylov subspace approximation error. We seek the error committed between the exact solution and solution of parabolic equations(heat equation) by Krylov approximation methods.The idea of the method is to approximate the action of the evolution operator on a given state vector by means projection process onto a Krylov subspace. This estimate will allow us not only to theoretically study the behavior of the convergence of the Krylov method as well as its stability but also allow us to give the exact size of the Krylov space according to the fixed stop test and the precisions Wish to establish.

***Keywords**:* Inverses Problems; heat Source; Krylov subspace; Matrix exponential; Krylov projection method; singularity of function; SVD method.

1. Introduction

From a partial knowledge of the solution u of a partial differential equation (internal measures, border measures), finding f identification of source is the problem of an inverse problem. The projection methods are algorithms for calculating eigenvalues and eigenvectors of matrix A of order n in a subspace K of dimension $m \ll n$ using a matrix H of order m of linear operator associated with the matrix A in K. If the matrix A is symmetric, the Lanczos method builds a symmetric tri-diagonal matrix H. If the matrix A is not symmetric, the Arnoldi method builds a matrix H upper Hessenberg. In both cases, if the algorithm reaches the mth $m \ll n$, we obtain a matrix H_m dupper Hessenberg form of size $m \times m$, and an orthonormal matrix Q_m $n \times m$, the m columns are defined by vectors $q_1, q_2, \cdots, q_m$. These vectors form an orthonormal basis of the Krylov subspace

$$K_m(A, q_1) = Vect\{q_1, Aq_1, A^2q_1, \cdots, A^{m-1}q_1\}.$$

Where $Vect\{\cdots\}$ denotes the set of linear combinations of the elements that lie between the hooks.

The algorithm's parameters:

- q_1, any initial vector,
- A, the matrix system,
- m, the size of the Krylov subspace.

We consider the following inverse problem: Find the pair of functions $(u(x,t), f(x))$ ***which satisfies: (cf.[1])***

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} = f(x) & 0 < x < 1,\ \ 0 < t \leq 1 \\ u(x,0) = 0, & 0 \leq x \leq 1 \\ \dfrac{\partial u}{\partial x}(0,t) = \dfrac{\partial u}{\partial x}(1,t) = 0, & 0 \leq t \leq 1 \\ u(x,1) = g(x), & 0 \leq x \leq 1. \end{cases} \tag{1}$$

$u(x,t)$ is the body temperature at a given point x of the axis at a given time t, and $f(x)$ is the unknown source of heat depending only on the spatial variable x.

This problem is called the inverse problem of identification of unknown source.

The boundary conditions:

$$\begin{cases} u(x,0) = 0, & 0 \leq x \leq 1 \\ \dfrac{\partial u}{\partial x}(0,t) = \dfrac{\partial u}{\partial x}(1,t) = 0, & 0 \leq t \leq 1 \end{cases} \tag{2}$$

The final condition: $u(x,1) = g(x)$, where g is a given measurement input internal. In applications, the input data $g(x)$ can only be measured, and there will be measured data function $g_\delta(x)$ which is merely in $L^2(0,1)$ and satisfies

$$\|g - g_\delta\|_{L^2(0,1)} \leq \delta \tag{3}$$

where the constant $\delta > 0$ represents a noise level of input data.

By the separation of variables, the solution of Problem (1) can be obtained as follows:

$$u(x,t) = \sum_{n=1}^{\infty} \frac{1-e^{-n^2\pi^2 t}}{n^2\pi^2} f_n e_n \tag{4}$$

where

$$\{e_n = \sqrt{2} cosn\pi x, (n = 1,2,\cdots)\} \tag{5}$$

is an orthogonal basis in $L^2(0,1)$, and

$$f_n = \sqrt{2}\int_0^1 f(x) cos(n\pi x) dx. \tag{6}$$

Making use of the final condition :

$$g(x) = \sum_{n=1}^{\infty} (g, e_n) e_n = \sum_{n=1}^{\infty} g_n e_n = \sum_{n=1}^{\infty} \frac{1-e^{-n^2\pi^2}}{n^2\pi^2} f_n e_n \tag{7}$$

and defining the operator $K : f \longrightarrow g$, we obtain:

$$g(x) = Kf(x) = \sum_{n=1}^{\infty} \frac{1-e^{-n^2\pi^2}}{n^2\pi^2} f_n e_n. \tag{8}$$

It is easy to see that K is a linear compact operator, and the singular values $\{\gamma_n\}_{n=1}^{\infty}$ of K are

$$\gamma_n = \frac{1-e^{-n^2\pi^2}}{n^2\pi^2}, (n = 1,2,\cdots). \tag{9}$$

On the other hand

$$g_n = (g, e_n) = \gamma_n f_n (e_n, e_n) \tag{10}$$

i.e.,

$$f_n = \gamma_n^{-1} g_n. \tag{11}$$

Therefore

$$f(x) = K^{-1} g(x) = \sum_{n=1}^{\infty} \frac{1}{\gamma_n} g_n e_n = \sum_{n=1}^{\infty} \frac{n^2\pi^2}{1-e^{-n^2\pi^2}} g_n e_n. \tag{12}$$

Note that $\frac{1}{\gamma_n} \longrightarrow \infty$ if $n \longrightarrow \infty$, which makes a small perturbation g cause the explosion of the solution. So, the problem is ill-posed because the solution does not continuously depend on the initial data. As there is no source of heat which is supplied indefinitely, we posed the question of the applicability of an effective method of truncation for the identification and regularization of the solution.

We propose in this work:

- In section 2, the applicability of the Krylov method for identifying the source heat, ant its regularizing effect.
- Section 3 is devoted to the error estimated by replacing the eigenvalues of the matrix A by the eigenvalues of the matrix H (Ritz values) for the efficiency of the method.
- And finally, in Section 4 we some remarks and conclusion.

2. Approaching Problem (1) by the Krylov Method

Let $H = L^2(0,1)$. We consider Problem (1): Find the pair of functions $(u(x,t), f(x))$ that satisfies

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} = f(x) & 0 < x < 1,\ 0 < t \leq 1 \\ u(x,0) = 0, & 0 \leq x \leq 1 \\ \dfrac{\partial u}{\partial x}(0,t) = \dfrac{\partial u}{\partial x}(1,t) = 0, & 0 \leq t \leq 1 \\ u(x,1) = g(x), & 0 \leq x \leq 1. \end{cases}$$

It is easy to see that the pair of functions

$$u(x,t), f(x)) = \left(\frac{1-e^{-\pi^2 t}}{\pi^2} cos(\pi x), cos(\pi x) \right) \tag{13}$$

is the exact solution of Problem (1.1). Consequently, the data function is $g(x) = \dfrac{1-e^{-\pi^2}}{\pi^2} cos(\pi x)$.

You can put the couple in the form of solution:

$$\big(u(x,t), f(x)\big) = \left((1-e^{-\pi^2 t}) cos(\pi x), \pi^2 cos(\pi x) \right) \tag{14}$$

and

$$g(x) = (1-e^{-\pi^2}) cos(\pi x). \tag{15}$$

2.1. *Appoximation*

Either the system $Au = v \Leftrightarrow u = \varphi(A) v$. Our goal is to obtain a solution approached this system that is sufficiently precise for the needs and lowest possible cost of calculation.

The heat source identified is given by equation (12) :

$$f(x) = K^{-1}g(x) = \sum_{n=1}^{+\infty} \frac{1}{\gamma_n} g_n e_n = \sum_{n=1}^{+\infty} \frac{n^2\pi^2}{1-e^{-n^2\pi^2}} g_n e_n.$$

Which can be written numerically $f(x) = \sum_{n=1}^{N} \varphi(\lambda_n) g_n e_n$, where $\varphi(s) = \frac{s}{1-e^{-s}}$, λ_n, and e_n are, respectively, the eigenvalues and eigenvectors of the matrix of the discretized operator A_h or $g(x) = \sum_{n=1}^{N} g_n e_n$. So, f is of the form

$$f = \varphi(A)g = (I_n - exp(-A))^{-1} A g \tag{16}$$

where $f, g \in \mathbb{R}^n$, $A \in M_n(\mathbb{R})$.

Let A be the unbounded operator defined by:

$$\begin{cases} \mathfrak{D}(A) = \{u \in H^1(0,1); u'(1) = u'(0) = 0\} \\ Au = -\dfrac{d^2u}{dx^2} \end{cases} \tag{17}$$

where $\mathfrak{D}(A)$ is the domain of definition of the operator A.

Proposition 0.1. *The operator A is self-adjoint and positive. (cf.[10])*

After the semi-discretization of the operator A, we have:

$$A_h = \frac{1}{h^2} \begin{pmatrix} 1 & -1 & 0 & 0 & . & . & 0 & 0 \\ -1 & 2 & -1 & 0 & . & . & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & . & . & 0 \\ . & 0 & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 0 & 0 & . & . & 0 & -1 & 2 & -1 \\ 0 & 0 & . & . & 0 & 0 & -1 & 1 \end{pmatrix} \tag{18}$$

The matrix A_h is tridiagonal and symmetric by construction.

Proposition 0.2. *The matrix A_h is symmetric and positive.*

2.2. *Arnoldi Algorithm*

(cf.[6]) Let $A \in M_n(\mathbb{R})$, $q_1 \in \mathbb{R}^n$, and $m \ll n$. This algorithm computes the factorization of A in the form $AQ = QH$, where $Q \in M_{n,m}(\mathbb{R})$ orthonormal columns and $H \in M_m(\mathbb{R})$ upper Hessenberg. That is to say, it calculates an orthogonal basis of a Krylov subspace generated by $(g, Ag, A^2g, ..., A^{m-1}g)$ by the method of Arnoldi and returns the stored column vectors in Q and the base of the Hessenberg matrix orthogonalization coefficients in H. It chooses an arbitrary vector q_1

Algorithm

1. $q_1 = \dfrac{g}{\|g\|_2}$
2. For $j = 1 : m$ do
3. $w := Aq_j$
4. For $i = 1 : j$ do

5. $h_{i,j} := q_i^* w$
6. $w := w - h_{i,j} q_i$ $\longrightarrow$ orthogonalization of w, $h_{i,j} = (Aq_j, q_i),\ i \leq j$
7. end
8. $h_{j+1,j} := \|w\|_2$ $\longrightarrow$ $h_{j+1,j} = 0$ if $j = L$
9. if $h_{j+1,j} > 0$ then
10. $q_{j+1} := \dfrac{w}{h_{j+1,j}}$
11. otherwise
12. $q_{j+1} := 0$

Arnoldi Approximation

Was as given: $A \in M_n(\mathbb{R})$, $g \in \mathbb{R}^n$, and $m \ll n$. The result of the approximation is given by

$$\varphi(A)g \approx f_m = \|g\| Q_m \varphi(H_m) e_1 \tag{19}$$

where Q_m and H_m are the matrix of the Arnoldi procedure, and e_1 is the first vector of the canonical basis of $\mathbb{R}^m$ (cf.[4]).

As the matrix A is tridiagonal and symmetric, for convenience, an algorithm is used which makes the Lanczos matrix $H_m = T_m$ a tridiagonal symmetric matrix, as the Arnoldi method is more general. (cf.[3])

2.3. *Lanczos Algorithm*

(cf.[9])

1. Let $q_0 = 0$, $\beta_0 = 0$, and $q_1 \in \mathbb{R}^N$ such that $\|q_1\|_2 = 1$
2. Is calculated successively for $j = 1, \cdots, m$
3. $z_j := Aq_j - \beta_j q_{j-1}$ $\longrightarrow$ Direct iteration
4. $\alpha_j := (z_j, q_j)$ $\longrightarrow$ Scalar Product
5. $z_j := z_j - \alpha_j q_j$ $\longrightarrow$ Orthogonalization
6. $\beta_{j+1} := \|z_j\|_2$ $\longrightarrow$ calculate the Euclidean norm
7. if $\beta_{j+1} := 0$, then stop.
8. $q_{j+1} := \dfrac{z_j}{\beta_{j+1}}$ $\longrightarrow$ Normalization
9. End

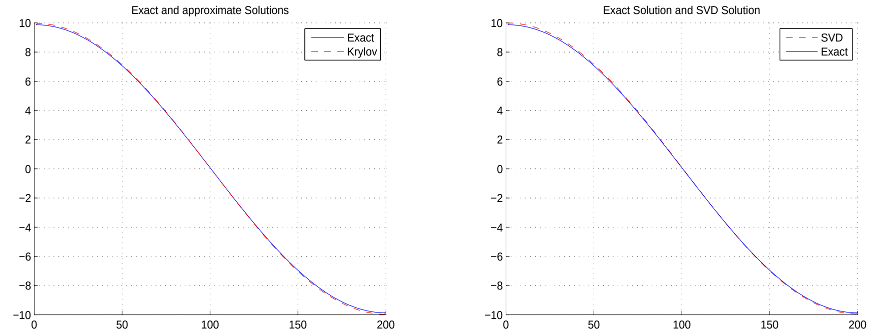
Is thus obtained m orthonormal vectors q_j and a tridiagonal symmetric matrix T size $m \times m$ (matrix Rayleigh) of diagonal elements
$\alpha_j = T_{j,j}$ and extra diagonal $\beta_j = T_{j,j-1} = T_{j-1,j}$.
The Krylov approximation of the vector f is given by:

$$f_m = \|g\| Q_m \varphi(T_m) e_1 \tag{20}$$

$$or \quad \varphi(T_m) = (I_m - exp(-T_m))^{-1} T_m.$$

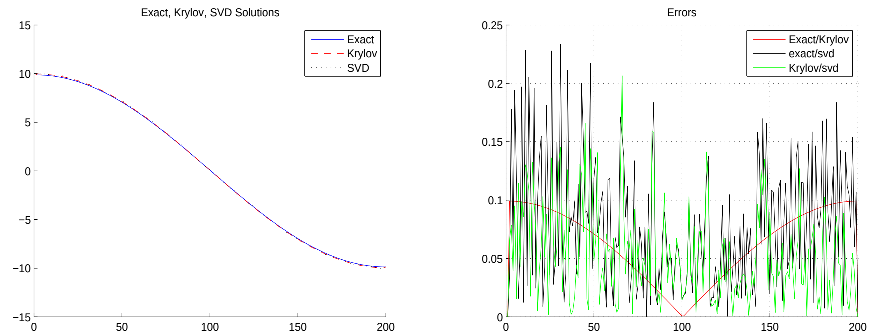
2.4. ***Numerical Results***

We took $N = 200$, $m = 100$, and—as the starting vector of the Lanczos Algorithm—$v = g$, with $g \in \mathbb{R}^N$, which is the test function.



a b

Figure 1: Graphical representation:(a) exact solution and approximation of Krylov, (b) exact solution and singular value decomposition (SVD) solution.



c d

Figure 2: Graphical representation:(c) exact solution, approximation of Krylov and SVD solution, (d) respective errors.

The SVD (Singular Value Decomposition)is the method used by Matlab to calculate the matrix functions.

3. Error Estimation

Let $A \in M_n(\mathbb{R})$ symmetric positive definite matrix. Our goal is to calculate the error estimate of $\|f - f_m\|_2$ to assess the accuracy of the result. Substituting the values of the matrix $\varphi(A)$ by those of $\varphi(H_m)$, what is the error?

Our objective here is to estimate the quantity $\|f - f_m\|$.

Let $A \in M_n(\mathbb{R})$ be a positive symmetric definite matrix. It has

$$f = \varphi(A)g = A(I - e^{-A})^{-1}g = \sum_{i=1}^{n} \frac{\lambda_i}{1 - e^{-\lambda_i}} g_i e_i = \sum_{i=1}^{n} \frac{\lambda_i}{1 - e^{-\lambda_i}} (g, e_i)e_i = \sum_{i=1}^{n} \varphi(\lambda_i)(g, e_i)e_i \tag{21}$$

$g \in \mathbb{R}^N$, where (λ_i, e_i) are the pair of eigenvalues and eigenvectors of the matrix A:

$$Ae_i = \lambda_i e_i, \ \ (e_i, e_j) = \delta_{ij}, \ \ \mathbb{R}^n = \bigoplus_{i=1}^{n} [e_i].$$

$$f_m = \|g\|_2 Q_m \varphi(H_m)e_1 = \|g\|_2 [Q_m H_m (I - e^{-H_m})^{-1}]e_1. \tag{22}$$

We use the Chebyshev series that are important approximation theory. It is known that the truncated expansion of a function by Chebyshev series polynomial is the best at approximating the sense of the uniform norm on the segment $[-1,1]$. (cf.[7])

Let $\lambda_1, \lambda_2, \cdots \lambda_n$ and $\beta_1, \beta_2, \cdots \beta_m$, with $m \ll n$ the respective eigenvalues of A, and $H = Q^t AQ$, where $H \in M_m(\mathbb{R})$ the Hessenberg matrix, $Q \in M_{n,m}(\mathbb{R})$, $Q^t Q = I$, $\|Q\| < 1$.

We have the following relation between the eigenvalues of A and those of H (Ritz values) (cf.[12]) :

$$\lambda_1 \leq \beta_i \leq \lambda_n, \quad i = 1, 2, \cdots m, \ \ with \ \ \lambda_{i+1} \geq \lambda_i, \ \ \beta_{i+1} \geq \beta_i.$$

If $g = \sum_{i=1}^{n} g_i e_i = \sum_{i=1}^{n} (g, e_i) e_i$, where (e_i) is an orthogonal basis of $L^2[0,1]$, then $f = \sum_{i=1}^{n} g\varphi(\lambda_i) e_i$. The (g, e_i) are called Fourier coefficients.

As approximate values of f we take the vector $f_m = p_m(A)g$, where p_m is a polynomial $deg(p_m) \leq m - 1$.

The λ_i and β_i are known. Let λ_1 and λ_n be lower and upper bounds, respectively, of the spectral interval of A. Let $B = \frac{\lambda_n + \lambda_1}{\lambda_n - \lambda_1} I_n - \frac{2}{\lambda_n - \lambda_1} A$, with $-I_n \leq B \leq I_n$ or else $\|B\| \leq 1$ and $y = \frac{1}{\lambda_n - \lambda_1}(\lambda_n + \lambda_1 - 2x)$ is a linear function defined over the spectral interval of A and I_n matrix of order n. Define a function h on the segment $[-1,1]$ so that $h(B) = \varphi(A)$:

$$h(x) = \varphi\left(\frac{\lambda_n + \lambda_1 - (\lambda_n - \lambda_1)x}{2}\right). \tag{23}$$

Consider the Chebyshev series of function h, $h(x) = \sum_{k=0}^{+\infty} h_k T_k(x)$, where h_k are the Chebyshev coefficients with $h_k = \frac{min(2, k+1)}{\pi} \int_1^{-1} h(x) T_k(x)(1 - x^2)^{-\frac{1}{2}}$

and T_k are Chebyshev polynomials of the first kind and satisfy the recurrence relation: $T_0(x) = 1$, $T_1(x) = x$, $T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x)$, $k \geq 1$.

The family (T_k) forms a basis of the Hilbert space $L^2(\omega)$, where $\omega = \frac{1}{\sqrt{1 - x^2}}$ on the $[-1,1]$ is the measurement of the dot product.

It is orthogonal with respect to the weight function ω:

$$\int_1^{-1} T_n(x) T_m(x) \omega(x) dx = 0 \quad for \ \ n \neq m.$$

As $f_m = p_m(A)g = \|g\| Qp(H_m) e_1$ because $(K_m \approx P_{m-1})$ you can also ask in the same way:

$$V = \frac{\lambda_n + \lambda_1}{\lambda_n - \lambda_1} I_n - \frac{2}{\lambda_n - \lambda_1} H, \quad \|V\| \leq 1. \tag{24}$$

Theorem 0.3. *(cf.[2]) Assume that the series* $h(x) = \sum_{k=0}^{+\infty} h_k T_k(x)$ *is absolutely convergent in* $[-1,1]$, *then hold the following equalities:*

$$f - f_m = \sum_{k=m}^{+\infty} \left(h_k T_k(B) g - h_k Q \|g\| T_k(V) e_1 \right) = \sum_{k=m}^{+\infty} h_k \left(T_k(B) g - \|g\| Q T_k(V) e_1 \right) \tag{25}$$

we have the error bound:

$$\|f - f_m\| \leq 2\|g\| \sum_{k=m}^{+\infty} |h_k| < +\infty. \tag{26}$$

Calculation of error $\theta_m = \|f - f_m\|$

For function $f = \varphi(A)g = A(I - e^{-A})^{-1} g$, we try to bound the Fourier–Chebyshev coefficients of the function linearly translated onto the spectral interval of the symmetric positive definite matrix A by estimating the function's values on ellipses whose poles are the endpoint of the spectral interval. An ellipse and its interior must not contain the function's singularities $(z = 0)$. We guess that the sum of semi-axes of such an ellipse can be optimized.

Let E_R be the ellipse in the complex plane with foci -1 and 1, and with the sum of semiaxes R, $R > 1$. This ellipse is also determined by the formula

$$E_R = \left\{ \frac{Re^{i\phi} + R^{-1}e^{-i\phi}}{2}\ ,\ 0 \leq \phi \leq 2\pi \right\}. \tag{27}$$

Let

$$a = \frac{\lambda_n - \lambda_1}{2} > 0, \qquad c = \frac{\lambda_n + \lambda_1}{\lambda_n - \lambda_1} > 1.$$
$$B = \frac{\lambda_n + \lambda_1}{\lambda_n - \lambda_1} I_n - \frac{2}{\lambda_n - \lambda_1} A, \quad \|B\| \leq 1.$$

We have:

$$[I - e^{-A}]^{-1} A = [I - e^{-a(cI_n - B)}]^{-1} a(cI_n - B) \tag{28}$$

because $h(B) = \varphi(A)$, then $h(\omega) = (1 - e^{-a(c-\omega)})^{-1} a(c - \omega)$, $\omega \in E_R$. We accept the following theorem

Theorem 0.4. *(cf.[8])*

If a function f is analytic in the domain, bounded by the ellipse E_R, and continuous up to E_R, such that

$$|f(z)| \leq M(R), z \in E_R,$$

then the Chebyshev coefficients a_k of f satisfy the inequality

$$|a_k| \leq 2M(R)R^{-k}$$

and for every $r \in [1, R)$ the Chebyshev series

$$\frac{a_0}{2} T_0(z) + \sum_{k=1}^{\infty} a_k T_k(z)$$

of the function f uniformly converges to f on the closed set bounded by the ellipse E_r.

Applying the theorem 0.4 to the function h. We have
$h(\omega) = (1 - e^{-a(c-\omega)})^{-1} a(c - \omega)$ with $\omega \in E_R$. It can be continuous extended:

$$\hat{h}(\omega) = \begin{cases} \dfrac{a(c-\omega)}{1 - e^{-a(c-\omega)}}, c \neq \omega \\ h(c) = 1, \end{cases} \tag{29}$$

where $\hat{h}$ is analytic at c. Therefore, c is a removable singularity of the function h. As the ellipse must not contain singularities of the function $\hat{h}$, seek other singularities in the plane complex near the spectral range. We know, in general, that the function

$$e^{x+iy} = e^z$$

is an entire function which has a single point at infinity. This is a periodic function of period of y 2π, and therefore a function of z $2\pi i$ period.

$$1 - e^{-a(c-\omega)} = 0 \Leftrightarrow e^{-a(c-\omega)} = 1 = e^{2\pi ni}, n \in \mathbb{Z}$$

Therefore, the function $\hat{h}$ has pole of order 1 to

$$-a(c - \omega) = 2\pi ni \quad for \quad n = \cdots, -1, 0, 1, \cdots.$$

The nearest singularities of the spectral interval are $n = -1, 0, 1$, and consequently, one has:

$$-a(c-\omega) = 2\pi ni \Leftrightarrow \omega = c + \frac{2\pi ni}{a}.$$

For $n = -1$, we have $\omega = c - \frac{2\pi i}{a}$

For $n = 0$, we have $\omega = c$

For $n = 1$, we have $\omega = c + \frac{2\pi i}{a}$

thus, $c + \frac{2\pi i}{a}$ *and* $c - \frac{2\pi i}{a}$ are non-removable singularities of $\hat{h}$.

For symmetric positive systems as defined in our case, the ellipse envelope spectrum of A degenerates into spectral interval $[\lambda_{min}, \lambda_{max}]$ on the positive part of the x-axis.

$$\omega \in E_R \Rightarrow \omega = \frac{Re^{i\phi} + R^{-1}e^{-i\phi}}{2}.$$

If we put the ellipse minor axis $\rho_1 = \dfrac{1}{2}(R + \dfrac{1}{R})$ and $\rho_2 = \dfrac{1}{2}(R - \dfrac{1}{R})$ the major axis, we have

$$\frac{1}{2}(R - \frac{1}{R}) \leq |\omega| \leq \frac{1}{2}(R + \frac{1}{R}). \tag{30}$$

Introduce the Zhukovsky function $\psi(z) = \dfrac{1}{2}(z + \dfrac{1}{z}), z \neq 0$, and its inverse is $\phi(\omega) = \omega + \sqrt{\omega^2 - 1}$, $\omega \in \mathbb{C} \setminus [-1, 1]$ where $\mathbb{C}$ is the set of complex numbers (so $\phi(\omega)$ is defined for all complex ω excepting those belonging to the real line segment $[-1, 1]$). The nearest singularities of $\hat{h}$ are on the ellipse E_R, with

$$R = \left|\phi\left(c \pm \frac{2\pi i}{a}\right)\right|. \tag{31}$$

The value of R is independent of the sign, due to the symmetry. Fix this value of R. Notice that these two non-removables singularities are simple poles. Then, decompose

$$\hat{h} = \hat{h}_1 + \hat{h}_2 + \hat{h}_3. \tag{32}$$

where

$$\hat{h}_1 = \hat{h} - \frac{Res(h, c + \frac{2\pi i}{a})}{\omega - c - \frac{2\pi i}{a}} - \frac{Res(h, c - \frac{2\pi i}{a})}{\omega - c + \frac{2\pi i}{a}}, \quad \hat{h}_2 = \frac{Res(h, c + \frac{2\pi i}{a})}{\omega - c - \frac{2\pi i}{a}}, \quad \hat{h}_3 = \frac{Res(h, c - \frac{2\pi i}{a})}{\omega - c + \frac{2\pi i}{a}}. \tag{33}$$

And Res denotes the residue of an analytic function at a pole.

Calculate R

$$R = \left|\phi\left(c + \frac{2\pi i}{a}\right)\right| = \left|c + \frac{2\pi i}{a} + \sqrt{\left(c + \frac{2\pi i}{a}\right)^2 - 1}\right| = \left|c + \frac{2\pi i}{a} + \sqrt{c^2 - \frac{4\pi^2}{a^2} + \frac{4i\pi c}{a} - 1}\right|.$$

Let $z = x + iy$, such that $c^2 - \frac{4\pi^2}{a^2} - 1 + \frac{4i\pi c}{a} = x^2 - y^2 + 2ixy$; we have:

$$\begin{cases} x^2 - y^2 = c^2 - \frac{4\pi^2}{a^2} - 1 & (i) \\ xy = \frac{2\pi c}{a} & (ii) \\ x^2 + y^2 = \sqrt{\left(c^2 - \frac{4\pi^2}{a^2} - 1\right)^2 + \frac{16\pi^2 c^2}{a^2}} & (iii) \end{cases}$$

$$(i) + (iii) \Rightarrow x^2 = \frac{1}{2}(c^2 - \frac{4\pi^2}{a^2} - 1) + \frac{1}{2}\sqrt{\left(c^2 - \frac{4\pi^2}{a^2} - 1\right)^2 + \frac{16\pi^2 c^2}{a^2}}$$

Letting x be positive for convenience, we have:

$$x = \left[\frac{1}{2}(c^2 - \frac{4\pi^2}{a^2} - 1) + \frac{1}{2}\sqrt{\left(c^2 - \frac{4\pi^2}{a^2} - 1\right)^2 + \frac{16\pi^2 c^2}{a^2}}\right]^{\frac{1}{2}}.$$

$$(ii) \Rightarrow y = \frac{2\pi c}{a} \times \frac{1}{y} = \frac{2\pi c}{a}\left[\frac{1}{2}(c^2 - \frac{4\pi^2}{a^2} - 1) + \frac{1}{2}\sqrt{\left(c^2 - \frac{4\pi^2}{a^2} - 1\right)^2 + \frac{16\pi^2 c^2}{a^2}}\right]^{-\frac{1}{2}}.$$

Thus, $z = x + iy =$

$$\left[\frac{1}{2}(c^2 - \frac{4\pi^2}{a^2} - 1) + \frac{1}{2}\sqrt{\left(c^2 - \frac{4\pi^2}{a^2} - 1\right)^2 + \frac{16\pi^2 c^2}{a^2}}\right]^{\frac{1}{2}} + i\frac{2\pi c}{a}\left[\frac{1}{2}(c^2 - \frac{4\pi^2}{a^2} - 1) + \frac{1}{2}\sqrt{\left(c^2 - \frac{4\pi^2}{a^2} - 1\right)^2 + \frac{16\pi^2 c^2}{a^2}}\right]^{-\frac{1}{2}}.$$

Let

$$\alpha = \left[\frac{1}{2}(c^2 - \frac{4\pi^2}{a^2} - 1) + \frac{1}{2}\sqrt{\left(c^2 - \frac{4\pi^2}{a^2} - 1\right)^2 + \frac{16\pi^2 c^2}{a^2}}\right]^{\frac{1}{2}}$$

and

$$\alpha^{-1} = \left[\frac{1}{2}(c^2 - \frac{4\pi^2}{a^2} - 1) + \frac{1}{2}\sqrt{\left(c^2 - \frac{4\pi^2}{a^2} - 1\right)^2 + \frac{16\pi^2 c^2}{a^2}}\right]^{-\frac{1}{2}}.$$

therefore

Whether

$$R = \sqrt{(c+\alpha)^2\left(1 + \frac{4\pi^2}{a^2\alpha^2}\right)}. \tag{34}$$

Calculate

$$\hat{h} = \hat{h}_1 + \hat{h}_2 + \hat{h}_3$$

The poles are simple:

$$\omega_n = c + \frac{2\pi n i}{a}, n \in \mathbb{Z}$$

Let $\omega = \omega_n + t,\ where\ t\ is\ \ a\ very\ \ small\ \ number.$

The residue of h at these points:

$$h(\omega_n + t) = \frac{a(\frac{-2n\pi}{a} - t)}{1 - e^{-a(\frac{-2n\pi i}{a} + t)}} = \frac{-at - 2n\pi i}{1 - e^{at}}. \tag{35}$$

By the division was:

$$h(\omega_n + t) = \frac{-2n\pi i - at}{-at - \frac{1}{2!}a^2t^2 - \frac{1}{3!}a^3t^3\cdots} = \frac{2n\pi i}{at} + \cdots.$$

The residuals are the coefficients of $\frac{1}{t} = a_{-1}$. Therefore, $Res(h, \omega_n) = \frac{2n\pi i}{a}$, and consequently,

$$Res(h, c + \frac{2\pi i}{a}) = \frac{2\pi i}{a},\ \ et\ \ Res(h, c - \frac{2\pi i}{a}) = \frac{-2\pi i}{a}.$$

Remark 0.5. As the poles are simple, we can use the formula $h(\omega) = \frac{f(\omega)}{g(\omega)}$.
If $f(a) \neq 0$, $g(a) = 0$, and $g'(a) \neq 0$, then $Res(\frac{f}{g}, a) = a_{-1} = \lim_{\omega \to a}(\omega - a)\frac{f(\omega)}{g(\omega)} = \frac{f(a)}{g'(a)}$.

$$\hat{h}_1 = \hat{h} - \frac{Res(h, c + \frac{2\pi i}{a})}{\omega - c - \frac{2\pi i}{a}} - \frac{Res(h, c - \frac{2\pi i}{a})}{\omega - c + \frac{2\pi i}{a}} = \frac{a(c-\omega)}{1 - e^{-a(c-\omega)}} - \frac{\frac{2\pi i}{a}}{\omega - c - \frac{2\pi i}{a}} + \frac{\frac{2\pi i}{a}}{\omega - c + \frac{2\pi i}{a}}.$$

$$\begin{aligned}\hat{h}_1(\omega) &= \frac{a(c-\omega)}{1 - e^{-a(c-\omega)}} - \frac{2\pi i}{a} \times \frac{1}{\omega - c - \frac{2\pi i}{a}} + \frac{2\pi i}{a} \times \frac{1}{\omega - c + \frac{2\pi i}{a}} \\ &= \frac{a(\omega - c)}{e^{a(\omega-c)} - 1} - \frac{2\pi i}{a(\omega - c - \frac{2\pi i}{a})} + \frac{2\pi i}{a(\omega - c + \frac{2\pi i}{a})}.\end{aligned} \tag{36}$$

Let us consider two discs $A_R(c - \frac{2\pi i}{a}, \frac{\pi}{a})$ and $B_R(c + \frac{2\pi i}{a}, \frac{\pi}{a})$ on the ellipse. We look for an increase of $\hat{h}_1(\omega)$ on the whole ellipse. We separately estimate the function on a set around the poles and on its complement in the ellipse. To do this, we will proceed in three steps.

Step 1

Let us consider the disc A_R with center $c - \frac{2\pi i}{a}$ and radius $\frac{\pi}{a}$

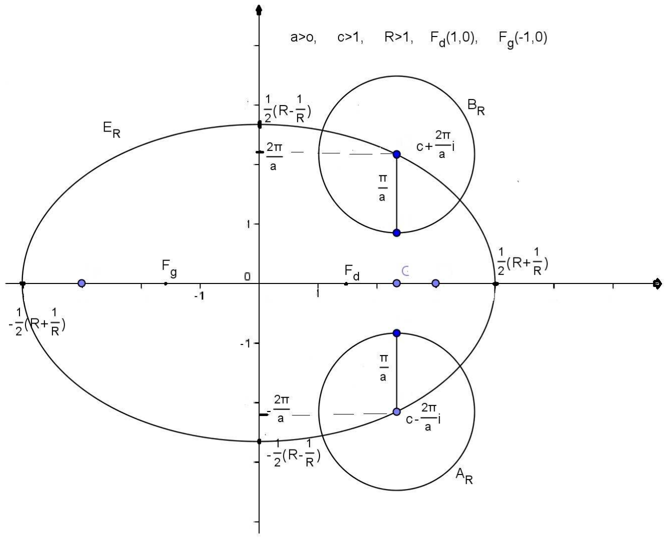


Figure 3: Error estimation around the poles of the ellipse.

$$A_R = \left\{ \omega \in E_R,\ |\omega - c + \frac{2\pi i}{a}| < \frac{\pi}{a} \right\}. \tag{37}$$

Let $X = a(\omega - c + \frac{2\pi i}{a}) = a(\omega - c) + 2\pi i$; therefore, $\omega = c - \frac{2\pi i}{a} + \frac{X}{a}$ and $\omega \in A_R \Rightarrow |X| < \pi$.

We have

$$F(X) = \hat{h}_1(\omega) = \hat{h}_1(c - \frac{2\pi i}{a} + \frac{X}{a}) = \frac{X - 2\pi i}{e^{X - 2\pi i} - 1} + \frac{2\pi i}{X} - \frac{2\pi i}{X - 4\pi i} = \frac{X - 2\pi i}{e^X - 1} + \frac{2\pi i}{X} - \frac{2\pi i}{X - 4\pi i}$$

$$= \frac{X}{e^X - 1} - \frac{2\pi i X}{X(e^X - 1)} + \frac{(2\pi i)X(e^X - 1)}{X^2(e^X - 1)} + \frac{1}{2}\frac{1}{(1 - \frac{X}{4\pi i})}$$

$$= \frac{X}{e^X - 1} - \frac{2\pi i X}{X(e^X - 1)} + \frac{(2\pi i)X(e^X - 1)}{X^2(e^X - 1)} + \frac{1}{2} \times \frac{1 - \frac{X}{4\pi i} + \frac{X}{4\pi i}}{(1 - \frac{X}{4\pi i})}$$

$$= \left[1 - \frac{2\pi i}{X} + \frac{(2\pi i)(e^X - 1)}{X^2} \right] \frac{X}{e^X - 1} + \frac{1}{2} + \frac{1}{8\pi i}\frac{X}{(1 - \frac{X}{4\pi i})}$$

We set $H(X) = \dfrac{X}{e^X - 1} = 1 - \dfrac{X}{2} + \displaystyle\sum_{n=1}^{+\infty} \frac{b_{2n}}{(2n)!} X^{2n}$, where b_{2n} are the numbers of Bernoulli.

It is known that

$$b_{2n}| \leq 4\frac{(2n)!}{(2\pi)^{2n}} \quad for \ \ n \geq 1 \tag{38}$$

By asking that $\Psi(X) = -\frac{X}{2} + \displaystyle\sum_{n=1}^{+\infty} \frac{b_{2n}}{(2n)!} X^{2n}$, we have $H(X) = 1 + \Psi(X)$, and $F(X)$ becomes

$$F(X) = \frac{1}{2} + \left[1 - \frac{2\pi i}{X} + \frac{(2\pi i)(e^X - 1)}{X^2} \right] (1 + \Psi(X)) + \frac{1}{8\pi i}\frac{X}{(1 - \frac{X}{4\pi i})}. \tag{39}$$

It is also known that

$$\frac{e^X - 1}{X^2} = \frac{1}{X^2}\left(\sum_{n=1}^{+\infty} \frac{X^n}{n!} \right) = \sum_{n=1}^{+\infty} \frac{X^{n-2}}{n!} = \frac{1}{X} + \frac{1}{2} + \sum_{n=3}^{+\infty} \frac{X^{n-2}}{n!} = \frac{1}{X} + \frac{1}{2} + \sum_{n=1}^{+\infty} \frac{X^n}{(n+2)!}$$

Calculate the term $1-\frac{2\pi i}{X}+\frac{(2\pi i)(e^X-1)}{X^2}$ in $F(X)$

$$1-\frac{2\pi i}{X}+\frac{(2\pi i)(e^X-1)}{X^2}=1-\frac{2\pi i}{X}+2\pi i\left(\frac{1}{X}+\frac{1}{2}+\sum_{n=1}^{+\infty}\frac{X^n}{(n+2)!}\right)=1+\pi i+2\pi i\sum_{n=1}^{+\infty}\frac{X^n}{(n+2)!}.$$

Then, $\frac{1}{1-\frac{X}{4\pi i}}=\sum_{n=0}^{+\infty}\left(\frac{X}{4\pi i}\right)^n$. Replace these terms by their value in $F(X)$

$$F(X)=\frac{1}{2}+\left[1+\pi i+2\pi i\sum_{n=1}^{+\infty}\frac{X^n}{(n+2)!}\right][1+\Psi(X)]+\frac{1}{8\pi i}X\sum_{n=0}^{+\infty}\left(\frac{X}{4\pi i}\right)^n$$

$$=\frac{1}{2}+1+\pi i+(1+\pi i)\Psi(X)+2\pi i(1+\Psi(X))\left(\sum_{n=1}^{+\infty}\frac{X^n}{(n+2)!}\right)+\frac{1}{8\pi i}X\sum_{n=0}^{+\infty}\left(\frac{X}{4\pi i}\right)^n$$

$$=\frac{3}{2}+\pi i+(1+\pi i)\Psi(X)+2\pi i(1+\Psi(X))\left(\sum_{n=1}^{+\infty}\frac{X^n}{(n+2)!}\right)+\frac{1}{8\pi i}X\sum_{n=0}^{+\infty}\left(\frac{X}{4\pi i}\right)^n.$$

By passing to the module we have

$$|F(X)|\leq\frac{\sqrt{9+4\pi^2}}{2}+\sqrt{1+\pi^2}|\Psi(X)|+2\pi(1+|\Psi(X)|)\left(\sum_{n=1}^{+\infty}\frac{|X|^n}{(n+2)!}\right)+\frac{1}{8\pi}|X|\sum_{n=0}^{+\infty}\left|\frac{X}{4\pi}\right|^n.$$

For $\omega\in A_R$, we have $|X|<\pi$; thus,

$$|\Psi(X)|\leq\frac{|X|}{2}+\sum_{n=1}^{+\infty}\left|\frac{b_{2n}}{(2n)!}\right||X|^{2n}\leq\frac{\pi}{2}+4\sum_{n=1}^{+\infty}\frac{1}{(2\pi)^{2n}}\pi^{2n}\quad according\ to\ the\ relation\ (38)$$

$$\frac{\pi}{2}+4\sum_{n=1}^{+\infty}\frac{1}{(2\pi)^{2n}}\pi^{2n}=\frac{\pi}{2}+4\sum_{n=1}^{+\infty}\left(\frac{1}{4}\right)^n$$

$$=\frac{\pi}{2}+\frac{4}{3}$$

$$\sum_{n=1}^{+\infty}\frac{|X|^n}{(n+2)!}<\sum_{n=1}^{+\infty}\frac{|X|^n}{n!}=e^{|X|}-1<e^\pi-1.$$

$$|X|\sum_{n=0}^{+\infty}\left|\frac{X}{4\pi}\right|^n<\pi\sum_{n=0}^{+\infty}\left(\frac{1}{4}\right)^n=\frac{4\pi}{3}.$$

And finally

$$|F(X)|\leq\frac{\sqrt{9+4\pi^2}}{2}+\sqrt{1+\pi^2}(\frac{\pi}{2}+\frac{4}{3})+2\pi(\frac{\pi}{2}+\frac{7}{3})(e^\pi-1)+\frac{1}{6}=M_1. \tag{40}$$

Let

$$\sup_{\omega\in A_R}|\hat{h}_1(\omega)|\leq M_1 \tag{41}$$

Step 2

Let us consider the disc B_R of center $c+\frac{2\pi i}{a}$ and radius $\frac{\pi}{a}$

$$B_R=\left\{\omega\in E_R,|\omega-c-\frac{2\pi i}{a}|<\frac{\pi}{a}\right\} \tag{42}$$

Let $Y=a(\omega-c-\frac{2\pi i}{a})=a(\omega-c)-2\pi i$; therefore, $\omega=c+\frac{2\pi i}{a}+\frac{Y}{a}$ and $\omega\in B_R\Rightarrow|Y|<\pi$.

We have

$$G(Y)=\hat{h}_1(\omega)=\hat{h}_1(c+\frac{2\pi i}{a}+\frac{Y}{a})=\frac{Y+2\pi i}{e^Y-1}-\frac{2\pi i}{Y}+\frac{2\pi i}{Y+4\pi i}$$

$$=\frac{Y}{e^Y-1}+\frac{2\pi iY}{Y(e^Y-1)}-\frac{(2\pi i)Y(e^Y-1)}{Y^2(e^Y-1)}+\frac{1}{2}\frac{1}{(1-\frac{Yi}{4\pi})}$$

$$
\begin{aligned}
G(Y) &= \frac{Y}{e^Y-1} + \frac{2\pi i Y}{Y(e^Y-1)} - \frac{(2\pi i)Y(e^Y-1)}{Y^2(e^Y-1)} + \frac{1}{2} \times \frac{1-\frac{Yi}{4\pi}+\frac{Yi}{4\pi}}{(1-\frac{Yi}{4\pi})} \\
&= \left[1+\frac{2\pi i}{Y} - \frac{(2\pi i)(e^Y-1)}{Y^2}\right]\frac{Y}{e^Y-1} + \frac{1}{2} + \frac{i}{8\pi}\frac{Y}{(1-\frac{Yi}{4\pi})} \\
&= \left[1+\frac{2\pi i}{Y} - \frac{(2\pi i)(e^Y-1)}{Y^2}\right][1+\Psi(Y)] + \frac{1}{2} + \frac{i}{8\pi}\frac{Y}{(1-\frac{Yi}{4\pi})}.
\end{aligned}
$$

Calculate the term $1+\frac{2\pi i}{Y} - \frac{(2\pi i)(e^Y-1)}{Y^2}$ in $G(Y)$

$$
\begin{aligned}
1+\frac{2\pi i}{Y} - \frac{(2\pi i)}{Y^2}(e^Y-1) &= 1+\frac{2\pi i}{Y} - \frac{(2\pi i)}{Y^2}\left(\sum_{n=1}^{+\infty}\frac{Y^n}{n!}\right) \\
&= 1+\frac{2\pi i}{Y} - (2\pi i)\left(\sum_{n=1}^{+\infty}\frac{Y^{n-2}}{n!}\right) \\
&= 1+\frac{2\pi i}{Y} - \frac{2\pi i}{Y} - \pi i - 2\pi i\left(\sum_{n=1}^{+\infty}\frac{Y^n}{(n+2)!}\right) \\
&= 1-\pi i - 2\pi i\left(\sum_{n=1}^{+\infty}\frac{Y^n}{(n+2)!}\right)
\end{aligned}
$$

We have:

$$
\begin{aligned}
G(Y) &= \frac{1}{2} + \left[1-\pi i - 2\pi i\sum_{n=1}^{+\infty}\frac{Y^n}{(n+2)!}\right][1+\Psi(Y)] + \frac{i}{8\pi}Y\sum_{n=0}^{+\infty}\left(\frac{iY}{4\pi}\right)^n \\
&= \frac{3}{2} - \pi i + (1-\pi i)\Psi(Y) - 2\pi i(1+\Psi(Y))\left(\sum_{n=1}^{+\infty}\frac{Y^n}{(n+2)!}\right) + \frac{i}{8\pi}Y\sum_{n=0}^{+\infty}\left(\frac{iY}{4\pi}\right)^n.
\end{aligned}
$$

Finally, we have:

$$
|G(Y)| \leq \frac{\sqrt{9+4\pi^2}}{2} + \sqrt{1+\pi^2}\left(\frac{\pi}{2}+\frac{4}{3}\right) + 2\pi\left(\frac{\pi}{2}+\frac{7}{3}\right)(e^{\pi}-1) + \frac{1}{6} = M_1. \tag{43}
$$

Let

$$
\sup_{\omega \in B_R} |\hat{h}_1(\omega)| \leq M_1. \tag{44}
$$

STEP 3

Let us consider $\omega \in E_R \setminus (A_R \cup B_R)$

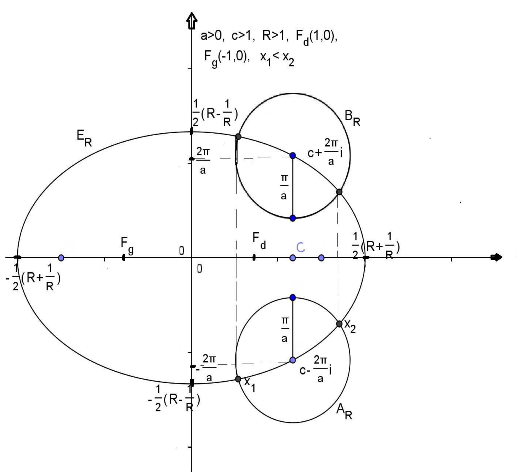


Figure 4: Error estimate on the complement of the set.

We have

$$\omega \in E_R \Rightarrow |\omega - c - \frac{2\pi i}{a}| \geq \frac{\pi}{a} \;\; and \;\; |\omega - c + \frac{2\pi i}{a}| \geq \frac{\pi}{a}. \tag{45}$$

Let the major axis be $\rho_1 = \frac{1}{2}(R + \frac{1}{R})$ and $\rho_2 = \frac{1}{2}(R - \frac{1}{R})$ the minor axis, and let us consider the circle of center c containing the ellipse.

We have

$$\forall \in E_R \setminus (A_R \cup B_R), |\omega - c| \leq \frac{1}{2}(R + \frac{1}{R}) + c = \rho_1 + c$$

$$\hat{h}_1(\omega) = \frac{a(\omega - c)}{e^{a(\omega - c)} - 1} - \frac{2\pi i}{a(\omega - c - \frac{2\pi i}{a})} + \frac{2\pi i}{a(\omega - c + \frac{2\pi i}{a})}$$

By passing to the module we have:

$$|\hat{h}_1(\omega)| = \frac{|a(\omega - c)|}{|e^{a(\omega - c)} - 1|} - \frac{2\pi}{a|\omega - c - \frac{2\pi i}{a}|} + \frac{2\pi}{a|\omega - c + \frac{2\pi i}{a}|} \leq \frac{a(\rho_1 + c)}{|e^{a(\omega - c)} - 1|} + \frac{2\pi}{a} \times \frac{a}{\pi} + \frac{2\pi}{a} \times \frac{a}{\pi}$$

$$|\hat{h}_1(\omega)| \leq 4 + \frac{a(\rho_1 + c)}{|e^{a(\omega - c)} - 1|.} \tag{46}$$

It is known that $|e^{a(\omega - c)} - 1| \geq ||e^{a(\omega - c)}| - 1|$; therefore,

$$|\hat{h}_1(\omega)| \leq 4 + \frac{a(\rho_1 + c)}{|e^{a(\omega - c)} - 1|} \leq 4 + \frac{a(\rho_1 + c)}{||e^{a(\omega - c)}| - 1|}.$$

Then, for $\omega \in E_R \setminus (A_R \cup B_R)$, take $\omega = x + iy$, and we have:

$$e^{a(\omega - c)} = e^{a(x - c + iy)} = e^{a(x - c)} \times e^{iay}, \qquad |e^{a(\omega - c)}| = e^{a(x - c)}.$$

A result

$$||e^{a(\omega - c)}| - 1| = |e^{a(x - c)} - 1|$$

. Seek $\mu > 0$, such that $\forall \omega \in E_R \setminus (A_R \cup B_R), \;\; ||e^{a(\omega - c)}| - 1| \geq \mu$.

Let x_1 , x_2 be the abscissa of the projection of two endpoints circles A_R and B_R on the major axis of the ellipse.

$$\omega = x + iy \Rightarrow x \leq x_1, \;\; where \;\; x \geq x_2.$$

Let $\omega_1 = (x_1 + iy_1)$,

$$\left|\omega_1 - c + \frac{2\pi i}{a}\right|^2 = \frac{\pi^2}{a^2} \Rightarrow (x_1 - c)^2 + (y_1 + \frac{2\pi i}{a})^2 = \frac{\pi^2}{a^2} \Rightarrow (x_1 - c)^2 \leq \frac{\pi^2}{a^2}$$

$$-\frac{\pi}{a} \leq x_1 - c \leq \frac{\pi}{a} \Rightarrow c - \frac{\pi}{a} \leq x_1 \leq c + \frac{\pi}{a}.$$

Which causes

$$c - \frac{\pi}{a} \leq x_1 < c \;\; and \;\; c < x_2 < c + \frac{\pi}{a}$$

then,

$$\begin{cases} a(x - c) \leq a(x_1 - c) < 0 \\ a(x - c) \geq a(x_2 - c) > 0 \end{cases} \Rightarrow \begin{cases} e^{a(c - x)} - 1 \leq e^{a(x_1 - c)} - 1 = e^{-a(c - x_1)} - 1 \\ e^{a(x_2 - c)} - 1 \geq e^{a(x - c)} - 1 \end{cases}$$

$$|e^{a(x - c)} - 1| \geq 1 - e^{-a(c - x_1)} \;\; for \;\; x < x_1$$

and

$$|e^{a(x - c)} - 1| \geq \min(1 - e^{-a(c - x_1)}, e^{a(x_2 - c)} - 1).$$

Let $\mu = \min(1-e^{-a(c-x_1)}, e^{a(x_2-c)}-1),\ \ 0<\mu<1$

$$x_1 \ and \ x_2$$

are the solution of

$$\begin{cases} \dfrac{x^2}{\rho_1^2}+\dfrac{y^2}{\rho_2^2}=1 \\ (x-c)^2+(y+\dfrac{2\pi}{a})^2=\dfrac{\pi^2}{a^2} \end{cases}$$

$$|\hat{h}_1(\omega)| \leq 4+\frac{a(\rho_1+c)}{\mu}. \tag{47}$$

Let

$$M(R)=\max\left(M_1, 4+\frac{a(\rho_1+c)}{\mu}\right) \tag{48}$$

where $\sup\limits_{\omega\in E_R} |\hat{h}_1(\omega)| \leq M(R)$.

It is a graph or a resolution to find the algebraic resolution abscissa x_1 and x_2 of the point of intersection E_R to the circle $(x-c)^2+(y+\dfrac{2\pi}{a})^2=\dfrac{\pi^2}{a^2}$ with $\mu=\min(1-e^{-a(c-x_1)}, e^{a(x_2-c)}-1)$ $0<\mu 1$.

By Theorem 0.4,

$$|h_{k,1}| \leq 2M(R)R^{-k} = 2\max\left(M_1, 4+\frac{a(\rho_1+c)}{\mu}\right)\times\left((c+\alpha)^2\left(1+\frac{4\pi^2}{a^2\alpha^2}\right)\right)^{\frac{-k}{2}}. \tag{49}$$

$$\hat{h}_2(\omega)=\frac{Res(h, c+\frac{2\pi i}{a})}{\omega-c-\frac{2\pi i}{a}}=\frac{\frac{2\pi i}{a}}{\omega-c-\frac{2\pi i}{a}}=\frac{-2\pi i}{a}\left(\frac{1}{(c+\frac{2\pi i}{a})-\omega}\right)$$

The Chebyshev coefficients of the functions $\hat{h}_2$ and $\hat{h}_3$ are determined by the identity

$$\frac{1}{z_0-\omega}=\frac{4\phi(z_0)^{-1}}{1-\phi(z_0)^{-2}}\sum_{k=0}^{\infty}\phi(z_0)^{-k}T_k(\omega),\ \ z_0\in\mathbb{C}\setminus[-1,1],$$

which is a reformulation of formula (25) in theorem 10.4 of Paszkowski's book (chapter 2, p.10). Apply this formula $\hat{h}_2$

$$\hat{h}_2(\omega)=\frac{2\pi i}{a}\left(\frac{1}{(c+\frac{2\pi i}{a})-\omega}\right)=\frac{2\pi i}{a}\left(\frac{4\phi(c+\frac{2\pi i}{a})^{-1}}{1-\phi(c+\frac{2\pi i}{a})^{-2}}\sum_{k=0}^{\infty}\phi\left(c+\frac{2\pi i}{a}\right)^{-k}T_k(\omega)\right).$$

Let $p=\phi(c+\frac{2\pi i}{a})^{-1}$; we have:

$$\hat{h}_2(\omega)=\frac{2\pi i}{a}\left(\frac{4p}{1-p^2}\sum_{k=0}^{\infty}p^kT_k(\omega)\right)=\sum_{k=0}^{\infty}\frac{2\pi i}{a}\left(\frac{4p}{1-p^2}p^k\right)T_k(\omega)=\sum_{k=0}^{\infty}\frac{8\pi i}{a}\left(\frac{p^{k+1}}{1-p^2}\right)T_k(\omega).$$

Or,

$$\hat{h}_2(\omega)=\sum_{k=0}^{\infty}h_kT_k(\omega).$$

So, by identification,

$$h_{k,2}=\frac{8\pi i}{a}\left(\frac{p^{k+1}}{1-p^2}\right).$$

By passing to the module we have:

$$|h_{k,2}|=\frac{8\pi}{a}\left|\frac{p^{k+1}}{1-p^2}\right| \tag{50}$$

Similarly,

$$\hat{h}_3(\omega)=\frac{-2\pi i}{a}\left(\frac{1}{(c-\frac{2\pi i}{a})-\omega}\right)=\frac{-2\pi i}{a}\left(\frac{4\phi(c-\frac{2\pi i}{a})^{-1}}{1-\phi(c-\frac{2\pi i}{a})^{-2}}\sum_{k=0}^{\infty}\phi\left(c-\frac{2\pi i}{a}\right)^{-k}T_k(\omega)\right).$$

Let $q = \phi(c - \frac{2\pi i}{a})^{-1}$; we have:

$$\hat{h}_3(\omega) = \frac{-2\pi i}{a}\left(\frac{4q}{1-q^2}\sum_{k=0}^{\infty} q^k T_k(\omega)\right) = \sum_{k=0}^{\infty}\frac{-2\pi i}{a}\left(\frac{4q}{1-q^2}q^k\right)T_k(\omega) = \sum_{k=0}^{\infty}\frac{-8\pi i}{a}\left(\frac{q^{k+1}}{1-q^2}\right)T_k(\omega).$$

Or,

$$\hat{h}_3(\omega) = \sum_{k=0}^{\infty} h_k T_k(\omega)$$

So, by identification,

$$h_{k,3} = \frac{-8\pi i}{a}\left(\frac{q^{k+1}}{1-q^2}\right)$$

By passing to the module we have:

$$|h_{k,3}| = \frac{8\pi}{a}\left|\frac{q^{k+1}}{1-q^2}\right|. \tag{51}$$

As a result,

$$|h_k| \le |h_{k,1}| + |h_{k,2}| + |h_{k,3}| = 2\max\left(M_1, 4 + \frac{a(\rho_1 + c)}{\mu}\right) \times \left((c+\alpha)^2\left(1 + \frac{4\pi^2}{a^2\alpha^2}\right)\right)^{\frac{-k}{2}} + \frac{8\pi}{a}\left|\frac{p^{k+1}}{1-p^2}\right| + \frac{8\pi}{a}\left|\frac{q^{k+1}}{1-q^2}\right|. \tag{52}$$

$$\sum_{k=m}^{+\infty}|h_k| \le 2\max\left(M_1, 4 + \frac{a(\rho_1 + c)}{\mu}\right) \times \sum_{k=m}^{+\infty} R^{-k} + \sum_{k=m}^{+\infty}\frac{8\pi}{a}\left|\frac{p^{k+1}}{1-p^2}\right| + \sum_{k=m}^{+\infty}\frac{8\pi}{a}\left|\frac{q^{k+1}}{1-q^2}\right|.$$

$$\sum_{k=m}^{+\infty} R^{-k} = \frac{R^{-m}}{1-R^{-1}}, \qquad \sum_{k=m}^{+\infty} p^{k+1} = \frac{p^{m+1}}{1-p}$$

$$\sum_{k=m}^{+\infty}\left|\frac{p^{k+1}}{1-p^2}\right| = \left|\frac{p^{m+1}}{(1+p)(1-p)^2}\right| = \left|\frac{\phi(c+\frac{2\pi i}{a})^{-m-1}}{(1+\phi(c+\frac{2\pi i}{a})^{-1})(1-\phi(c+\frac{2\pi i}{a})^{-1})^2}\right|$$

Similarly,

$$\sum_{k=m}^{+\infty}\left|\frac{q^{k+1}}{1-q^2}\right| = \left|\frac{q^{m+1}}{(1+q)(1-q)^2}\right| = \left|\frac{\phi(c-\frac{2\pi i}{a})^{-m-1}}{(1+\phi(c-\frac{2\pi i}{a})^{-1})(1-\phi(c-\frac{2\pi i}{a})^{-1})^2}\right|$$

Finally, by Theorem 26 we have the error estimate :

$$\theta_m = \|f - f_m\| \le 2\|g\|\sum_{k=m}^{+\infty}|h_k|$$

$$\theta_m \le 2\|g\|\left(2\max\left(M_1, 4 + \frac{a(\rho_1 + c)}{\mu}\right) \times \frac{\left((c+\alpha)^2\left(1+\frac{4\pi^2}{a^2\alpha^2}\right)\right)^{\frac{-m}{2}}}{1 - \left((c+\alpha)^2\left(1+\frac{4\pi^2}{a^2\alpha^2}\right)\right)^{\frac{-1}{2}}} + \right.$$
$$\frac{8\pi}{a}\left|\frac{\phi(c+\frac{2\pi i}{a})^{-m-1}}{(1+\phi(c+\frac{2\pi i}{a})^{-1})(1-\phi(c+\frac{2\pi i}{a})^{-1})^2}\right| +$$
$$\left.\frac{8\pi}{a}\left|\frac{\phi(c-\frac{2\pi i}{a})^{-m-1}}{(1+\phi(c-\frac{2\pi i}{a})^{-1})(1-\phi(c-\frac{2\pi i}{a})^{-1})^2}\right|\right)$$

These singularities will limit the convergence because they are poles of the ellipse, and it is understood that a singularity at a point on the ellipse has the same rate of asymptotic convergence a pole or other point of the ellipse.

4. Discussion

We studied the possibility of using a matrix algebra technique associated with a function which is a Krylov subspace technique. The analysis of the technique proposed in this work shows that the projection method can constitute an efficient tool for the approximation of an operator in the resolution of a system. In estimating the error, we theoretically study the behavior of the convergence of the Krylov method and its stability. This estimate allows us to give the exact size of the Krylov space by stopping the set of tests and the desired details. With our approach, it is now in particular possible to efficiently obtain upper bounds for the error by introducing the notions of residue and singularity of a function.

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